Lecture 7, 9/16/21
Material corresponds to Ross §13.

Sequences in a metric space

Let S be a metric space with distance function d. For a subset $E \subseteq S$ we say a sequence (s_n) "is in E" if $s_n \in E$ for all $n \in \mathbb{N}$.

Theorem A convergent sequence in S is Cauchy.

> **Definition** A metric space is **complete** if all Cauchy sequences converge.

Notation $x \in \mathbb{R}^k$, $x = (x_1, ..., x_k)$, $x_i \in \mathbb{R}$. We denote different elements of $\mathbb{R}^k$ with superscripts; $x^{(1)}, x^{(2)} \in \mathbb{R}^k$, $x^{(1)} = (x_1^{(1)}, ..., x_k^{(1)})$, $x^{(2)} = (x_1^{(2)}, ..., x_k^{(2)})$. A sequence in $\mathbb{R}^k$ is denoted $(x^{(n)})$, and the sequence formes by the i^{th} entry of $x^{(n)}$ is denoted $(x_i^{(n)})$. $\mathbb{R}^k$ is a metric space with distance funciton

$$d(x^{(1)}, x^{(2)})\sqrt{(x_1^{(1)} - x_1^{(2)})^2 + ... + (x_k^{(1)} - x_k^{(2)})^2}.$$

Theorem

$$x^{(n)} \to x \text{ if and only if } x_i^{(n)} \to x_i \text{ for all } i = 1, ..., k.$$

$$x^{(n)} \text{ is Cauchy if and only if } x_i^{(n)} \text{ is Cauchy forall } i = 1, ..., k.$$

Corollary $\mathbb{R}^k$ is a complete metric space.

Definition $T \subset S$ is bounded if there exists $s \in S$, $r \in \mathbb{R}$, such that for all $t \in T$ $d(s, t) < r$.

> **Theorem** (Bolzano-Weierstrass) A bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Open and Closed sets

Definition

- A set $E \subset S$ is open if for all $s_0 \in E$ there exists $r \in \mathbb{R}, r > 0$ such that $\{s \in S | \ d(s, s_0) < r\} \subset E$.
- A set E is closed if its complement $S \backslash E = \{s \in S | s \notin E\}$ is open.

Theorem A set $E \subset S$ is closed if and only if for every sequence (s_n) in E, $s_n \to s$ implies $s \in E$.

Compactness

Definition A set $E \subset S$ is **sequentially compact** if every sequence (s_n) in E has a convergent subsequence.

Theorem A subset of $\mathbb{R}^k$ is sequentially compact if and only if it is closed and bounded.