$(s_n)_n > . \mathbb{N}$ 3

limsup $-\varepsilon$ limsup v_N limsup $+\varepsilon$

Lemma Let (s_n) be a bounded sequence, $v = \limsup(s_n)$

3) Let $\varepsilon > 0$. Then $\{m \mid |s_m - \limsup s_n| < \varepsilon\}$ is infinite

Thm (Bolzano Weierstrass) Every bounded sequence contains a convergent subsequence

v , s_n

$n_1 \in \{m \mid |s_m - v| < 1\}$ (infinite)

$\wedge$

$n_2 \in \{n \mid |s_n - v| < 1/2\}$ (infinite)

$\wedge$

$\vdots$

$\wedge$

$n_k \in \{n \mid |s_n - v| < 1/k\}$ (infinite)

Continue for all k. Let $t_k = s_{n_k}$. We claim $t_k \to v$

Let $\varepsilon > 0$. Choose $N = 1/\varepsilon$. If $k > N$, then

$|t_k - v| = |s_{n_k} - v| < \frac{1}{k} < \frac{1}{N} = \varepsilon$. $\boxtimes$

Given a statement P, "(s_n) is in P" if $s_n \in P$ for all $n \in \mathbb{N}$.

Thm A convergent sequence is Cauchy in any metric space. (A)

Def A metric space is complete if all Cauchy sequences converge.

(so $\mathbb{R}$ is complete, we used the completeness axiom)

Example

$(0,1) \subset \mathbb{R}$ is a metric space with the same distance function $d(a,b) = |a-b|$.

$\frac{1}{n} \in (0,1)$ for all $n \in \mathbb{N}$, $(1/n)$ is Cauchy, $1/n \not\to s$ for any $s \in (0,1)$.

Notation $x \in \mathbb{R}^k$, $x = (x_1, \dots, x_k)$. If we have a lot of points,

$x^{(1)} = (x_1^{(1)}, \dots, x_k^{(1)})$, $x^{(2)}$, etc. A sequence in $\mathbb{R}^k$ is $(x^{(n)})$ for $n \in \mathbb{N}$

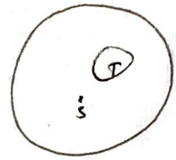
$d(x^{(1)}, x^{(2)}) = \sqrt{(x_1^{(1)} - x_1^{(2)})^2 + \dots + (x_k^{(1)} - x_k^{(2)})^2}$.

$\mathbb{R}^k$ is a metric space w/ d.

Theorem

$x^{(n)} \to x$ if and only if $x_i^{(n)} \to x_i$ for $i = 1, \dots, k$

$x^{(n)}$ is Cauchy if and only if $(x_i^{(n)})$ is Cauchy for $i = 1, \dots, k$.



<u>Corollary</u> $\mathbb{R}^k$ is a complete metric space.
$(x^{(n)})$ is Cauchy $\Rightarrow (x_i^{(n)})$ is Cauchy $\Rightarrow (x_i^{(n)})$ converges $\Rightarrow (x^{(n)})$ conv.

<u>Def</u> S a metric space with funct d, $T \subset S$ is <u>bounded</u> if there exists $s \in S$, $r \in \mathbb{R}$ such that for all $t \in T$, $d(s,t) < r$
(check this matches <u>bounded</u> in $\mathbb{R}$)

A sequence $(x^{(n)})$ is bounded in $\mathbb{R}^n$ if and only if $(x_i^{(n)})$ is bounded.
$\forall i \ldots n$

<u>Thm (Bolzano-Weierstrass)</u> A bounded sequence in $\mathbb{R}^n$ has a convergent subsequence

<u>Proof</u>: replace $(x^{(n)})$ w/ a subsequence such that $(x_1^{(n)})$ converges
replace $(x^{(n)})$ w/ a subseq s.t. $(x_2^{(n)})$ conv.
the corresponding subseq of $(x_1^{(n)})$ still converges

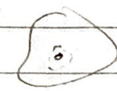
$\vdots$

repeat k times, $(x_i^{(n)})$ conv, so $(x^{(n)})$ converges

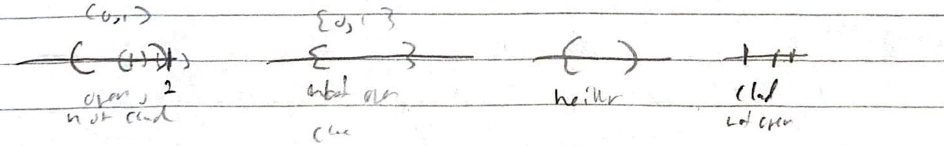
Let S be a metric space w/ distance d.

A set $E \subset S$ is <u>open</u> if for all $s_0 \in E$, there exists $r > 0$ such that $\{s \in S \mid d(s, s_0) < r\} \subset E$ (if $d(s, s_0) < r$ then $s \in E$).

A set $E \subset S$ is <u>closed</u> if $S \setminus E$ is open.



<u>Example</u>



A union of open sets is open. A intersection of a finite number of open sets is open.
An intersection of closed sets is closed. A union of a <u>finite</u> number of closed sets is closed.

Theorem A set $E \subset S$ is closed if and only if for all (s_n) in E, $s_n \to s$, $s \in S$, then $s \in E$

Compactness Let S be a metric space with distance d.

Def A subset $E \subset S$ is sequentially compact if every sequence (s_n) in E has a subsequence which converges to an element of E

Theorem A subset $E \subset \mathbb{R}^k$ is sequentially compact if and only if it is closed and bounded

Proof ($\Leftarrow$) Let E be closed and bounded. Let (s_n) is a sequence in E, then (s_n) is bounded. By BW, (s_n) has a convergent subseq $t_n \to s$. So (t_n) is in E, and E is closed, $s \in E$.

($\Rightarrow$) A sequentially compact set is closed:
Let E be sequentially compact. Let (s_n) be a seq in E and let $s_n \to s \in S$. Then (s_n) contains a subsequence converging to an element of E. But any subsequence converges to s, so $s \in E$. Thus E is closed

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A sequentially compact set is bounded:

Suppose E is not bounded. Let $t \in S$. Then for each $n \in \mathbb{N}$, $E \not\subset \{s \in S \mid d(s,t) < n\}$. So there exists $s_n \in E$ such that $d(s_n, t) \geq n$. The seq (s_n) has no Cauchy subsequence! Let (t_k) be a subseq, $t_k = s_{n_k}$.

$t_k = s_{n_k}$ $N \in \mathbb{N}$, $k > N$, $\ell > \max\{d(s_{n_\ell}, t)+1, N\}$

$$d(t_k, t_\ell) = d(s_{n_k}, s_{n_\ell}) \geq d(s_{n_k}, t) - d(s_{n_\ell}, t)$$
$$\geq n_k - d(s_{n_\ell}, t)$$
$$\geq k - d(s_{n_\ell}, t) \geq 1.$$

So (t_k) is not Cauchy, fails the $\varepsilon = 1$.