Useful facts

1) $|t-s| < \varepsilon \iff t \in (s-\varepsilon, s+\varepsilon)$
2) if $s \le t+\varepsilon$ for all $\varepsilon > 0$, then $s \le t$
3) If $s_n \le a$ and $s_n \to s$ then $s \le a$, for all $n \in \mathbb{N}$.

Definition A sequence (s_n) is bounded if $\{s_n\}$ is bounded.

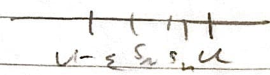
A sequence (s_n) is nondecreasing if $s_n \le s_{n+1}$ for all $n \in \mathbb{N}$
A sequence (s_n) is nonincreasing if $s_n \ge s_{n+1}$ for all $n \in \mathbb{N}$

A sequence is monotone if it is nonincreasing or nondecreasing

Thm A bounded monotone sequence converges

Proof
Let (s_n) be a bounded nondecreasing sequence.
Since $\{s_n\}$ is bounded, $u = \sup\{s_n\}$ exists.
(We want to show that $s_n \to u$)

Let $\varepsilon > 0$. Since $u - \varepsilon < u$, there exists $N \in \mathbb{N}$
such that $u - \varepsilon < s_N$.
If $n > N$, then since (s_n) is nondecreasing,



$$\Rightarrow u-\varepsilon < s_N \le s_n \le u.$$

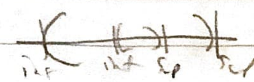
Thus for $n > N$, $|u - s_n| < \varepsilon$. and $s_n \to u$. ∎

Definition of Lim Sup / Lim inf

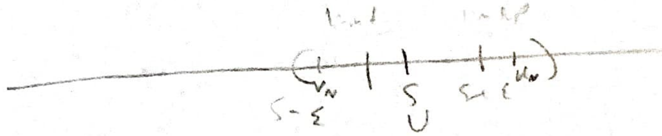
Let (s_n) be a bounded sequence, so
$m \le s_n \le M$ for all $n \in \mathbb{N}$.
"tail"
"N tail"
Thus for all $N \in \mathbb{N}$ the set $\{s_n \mid n > N\}$ is bounded and we can define

$$m \le \underline{u_N =} \inf\{s_n \mid n > N\} \le \sup\{s_n \mid n > N\} \underline{=} v_N \le M$$

Since $\{s_n \mid n > N+1\} \subseteq \{s_n \mid n > N\}$



$u_N \le u_{N+1}$ $\qquad v_{N+1} \le v_N$



$\cup$
$\{s_n | n>N\}$, so inf/sup in here

So (u_N) is non decreasing and (v_N) is nonincreasing, both are bounded

Def

$$\limsup s_n = \lim v_N = \lim_{N\to\infty} \sup\{s_n | n>N\}$$

$$\liminf s_n = \lim u_N = \lim_{N\to\infty} \inf\{s_n | n>N\}$$

Since $u_N \le v_N$ for all $N \in \mathbb{N}$, $\liminf s_n \le \limsup s_n$
($u_N - v_N \le 0$, so $\lim(u_N - v_N) \le 0$)

Thm Let (s_n) be a bounded sequence in $\mathbb{R}$. s_n converges if and only if $\liminf s_n = \limsup s_n$

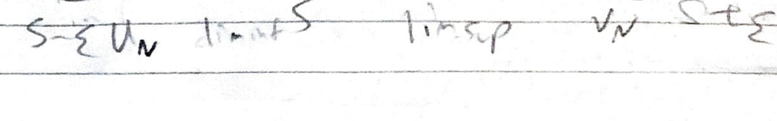
If s_n converges, then $\lim s_n = \liminf s_n = \limsup s_n$

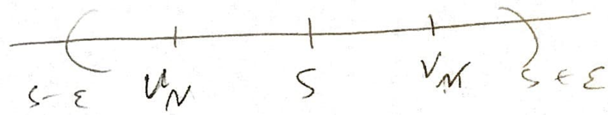
Proof

Step 1 ($\Rightarrow$) Assume $s_n \to s$ for some $s \in \mathbb{R}$.
(we want to prove $\liminf s_n = \limsup s_n$)

Let $\varepsilon > 0$
There exists $N \in \mathbb{R}$ such that for $n > N$, $|s_n - s| < \varepsilon$.
Then $s+\varepsilon$ is a upper bound for $\{s_n | n>N\}$
and $\sup\{s_n | n>N\} = v_N \le s+\varepsilon$.

* Thus $\limsup s_n = \lim v_N \le v_N \le s+\varepsilon$.
This is true for all $\varepsilon > 0$, so $\limsup s_n \le s$
Similar argument shows $\liminf s_n \ge s$.
Since $\liminf s_n \le \limsup s_n$, all three are equal





Step 2 ($\Leftarrow$) Assume $\liminf s_n = \limsup s_n$, call them s.

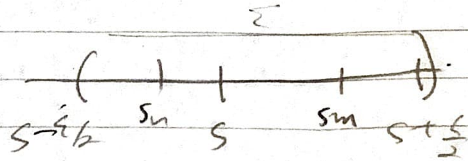
Let $\varepsilon > 0$. Let $v_m = \sup\{s_n \mid n > m\}$; then $v_m \to s$.
So Let $u_m = \inf\{s_n \mid n > m\}$ then $u_m \to s$.
So there exists N_1 such that for $m > N_1$, $|v_m - s| < \varepsilon$ and $v_m < s + \varepsilon$
N_2 such that for $m > N_2$, $|u_m - s| < \varepsilon$ and $u_m > s - \varepsilon$.
Choose $N > \max\{N_1, N_2\}$. Then for $n > N$, $s_n \in \{s_n \mid n > N\}$
and so $s - \varepsilon < u_N \le s_n \le v_N < s + \varepsilon$ so $|s_n - s| < \varepsilon$.

Lemma A Cauchy seq is bounded.
Let $\varepsilon = 1$. Then exists $N \in \mathbb{N}$ such that for $n, m > N$, $|s_n - s_m| < 1$.
Fix $n_0 > N$ and let $M = \max\{s_1, \dots, s_N, s_{n_0} + 1\}$
$m = \min\{s_1, \dots, s_N, s_{n_0} - 1\}$.
Then if $n \le N$, $m \le s_n \le M$ and if $n > N$, $m \le s_{n_0} - 1 < s_n < s_{n_0} + 1 \le M$.

Def A sequence s_n is Cauchy if
for all $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that
if $n, m > N$ then $|s_n - s_m| < \varepsilon$.

Theorem A sequence (s_n) in $\mathbb{R}$ converges if and only if it is Cauchy.

Step 1 ($\Rightarrow$) Assume $s_n \to s \in \mathbb{R}$. (want to show s_n is Cauchy)
Let $\varepsilon > 0$. Then exists $N \in \mathbb{R}$ such that for $n > N$, $|s_n - s| < \frac{\varepsilon}{2}$.
Let $n, m > N$. Then
$|s_n - s_m| \le |s_n - s| + |s - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.
So s_n is Cauchy



Step 2 ($\Leftarrow$) Assume s_n is Cauchy.

Let $\varepsilon > 0$. There exists $N \in \mathbb{R}$ such that if $n, m > N$ then $|s_n - s_m| < \varepsilon/2$. Fix $m > N$. Then for all $n > N$, $s_n \leq s_m + \varepsilon/2$, so $s_m + \varepsilon/2$ is an upper bound for $\{s_n \mid n > m\}$. Since s_n is bounded, $v_m = \sup\{s_n \mid n > m\}$ exists, and $v_m \leq s_m + \frac{\varepsilon}{2}$. Similarly, $\inf\{s_n \mid n > m\} = u_m \geq s_m - \varepsilon/2$. Thus

$$s_m - \varepsilon/2 \leq u_m \leq \liminf s_n \leq \limsup s_n \leq v_m \leq s_m + \varepsilon/2$$

and $|\limsup s_n - \liminf s_n| \leq \varepsilon$. Since this is true for all $\varepsilon > 0$, $\limsup s_n = \liminf s_n$ and s_n converges.

Complete

Completeness axiom (Least upper bound property)

monotone bounded seq conv $\longrightarrow$ Cauchy seq conv

limsup liminf

[illegible]