03 Let S be a metric space with distance function d.
2 Let $(s_n) = (s_1, s_2, \dots)$ be a sequence with $s_n \in S$.

<u>Def</u> $s_n \to s \in S$ if for all $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ there exists $N \in \mathbb{R}$ such that $n > N$ implies $d(s_n, s) < \varepsilon$.

<u>Example</u> $s_n = \frac{(-1)^n}{n} \to 0$

3 <u>Proof</u> Let $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. If $n > N$, then

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

<u>Example</u> $s_n = n$ does not converge to any $r \in \mathbb{R}$

<u>Pf</u> Let $r \in \mathbb{R}$. $\boxed{\text{Choose } \varepsilon = 1.}$ (1) We need to show that <u>no</u> N works for this ε. $\boxed{\text{Let } N \in \mathbb{R}.}$ (2) Choose $\boxed{n > \max\{N, r+1\}}$ (3)
Then $n > N$, but

$$|n - r| \ge n - r > r + 1 - r = 1 = \varepsilon$$

Process to prove s_n <u>does</u> <u>not</u> <u>converge</u> to $r \in \mathbb{R}$:

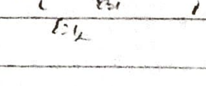
(1) The definition of converge is "for all $\varepsilon > 0$" so need <u>just one</u> ε that does not work.

(2) For ε to not work, there must be <u>no</u> N which works. So we need to check all $N \in \mathbb{R}$

(3) For N to not work, we need ~~one~~ $n > N$ s.t. $|s_n - s| \ge \varepsilon$

<u>Ex</u> $s_n = (-1)^n$ does not converge to any $r \in \mathbb{R}$.

Idea! Nothing is within ε of both 1 and -1



Proof Let $r \in \mathbb{R}$. Choose $\varepsilon = 1$ ① | Let $N \in \mathbb{N}$ ②

$|1-r| + |r-(-1)| \geq |1-(-1)| = 2$ ($\triangle$ ineq)

so either $|1-r| \geq 1$ choose even $n > N$

or $|(-1)-r| \geq 1$ choose odd $n > N$

either way $|s_n - r| = |(-1)^n - r| \geq 1 = \varepsilon$.

Limits are unique!

Thm If $s_n \to s$ and $s_n \to t$ then $s = t$

Proof (Idea: if s_n is very close to s, t, then s, t are close together)

Let $\varepsilon > 0$. Then there exists N_1 such that for $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2}$

There exists N_2 such that for $n > N_2$, $|s_n - t| < \frac{\varepsilon}{2}$.

Choose $n > \max\{N_1, N_2\}$. Then

$|s - t| = |\quad| \leq |s - s_n| + |s_n - t|$ (by $\triangle$ ineq)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

So $0 \leq |s-t| < \varepsilon$ for all $\varepsilon > 0$, and then $|s-t| = 0$ so $s = t$.

Example Show that $s_n = \dfrac{2n^2 - 7n}{n^2 - n + 8} \to 2$

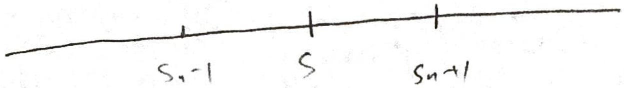
Idea: make n big enough that $\left|\dfrac{2n^2-7n}{n^2-n+8} - 2\right|$ is small, $< \varepsilon$

Approximate ~~that~~ quantity w/ a simple expression in n, figure out how big n must be so that the expr ~~less~~ $\leq \varepsilon$

$$\left|\frac{2n^2 - 7n - 2(n^2-n+8)}{n^2-n+8}\right| = \left|\frac{-5n-16}{n^2-n+8}\right| = \left|\frac{5n+16}{n^2-n+8}\right| \leq \left|\frac{5n+16}{n^2-n}\right|$$

$n > 1$ $\quad 16n \geq 16$ $\quad n - 1 \geq n/2$ if $\frac{n}{2} \geq 1$, $n \geq 2$

$$\leq \left|\frac{5n + 16n}{n^2 - n}\right| = \frac{21}{n-1} \leq \frac{21}{n/2} = \frac{42}{n} < \varepsilon \text{ if } n > \frac{42}{\varepsilon}.$$



Proof

Let $\varepsilon > 0$. Choose $N = \max\left\{\frac{42}{\varepsilon}, 2\right\}$. Let $n > N$.

Then $|s_n - 2| = \left|\frac{5n+16}{n^2-n-8}\right| \le \left|\frac{5n+16}{n^2-n}\right|$.

Since $n \ge 2$, $16n \ge 16$, so

$$\left|\frac{5n+16}{n^2-n}\right| \le \left|\frac{21n}{n^2-n}\right| = \left|\frac{21}{n-1}\right|.$$

Since $n \ge 2 > 0$, $\frac{n}{2} - 1 \ge 0$, and $n - 1 \ge n/2$.

Then $\left|\frac{21}{n-1}\right| \le \frac{42}{n} < \frac{42}{N} \le \frac{42}{(42/\varepsilon)} = \varepsilon$. So $|s_n - 2| < \varepsilon$. ∎

Limit Theorems   Let $s_n \to s$, $t_n \to t$. ($s_n, t_n, s, t \in \mathbb{R}$)

1) for all $k \in \mathbb{R}$, $ks_n \to ks$
2) $s_n + t_n \to s + t$
3) $s_n t_n \to st$
4) If $s_n \ne 0$ for all n and $s \ne 0$, then $1/s_n \to 1/s$ and $t_n/s_n \to t/s$

Prove that $\frac{2n^2 - 7n}{n^2-n-8} \to 2$. Since $n > 0$, $\frac{2n^2-7n}{n^2-n-8} = \frac{2 - \frac{7}{n}}{1 - \frac{1}{n} + \frac{8}{n^2}}$.

$\frac{1}{n} \to 0$, so $-7/n \to 0$ and $2 - 7/n \to 2$. For $n > 1$, $1 - \frac{1}{n} + \frac{8}{n^2} > 0$.
$\frac{1}{n} \to 0$, $\frac{8}{n^2} \to 0$, so $1 - \frac{1}{n} - \frac{8}{n^2} \to 1$. Then $2n^2 - 7n/n^2 - n - 8 \to 2/1 = 2$.

Theorem   If $s_n \to s$, then $\{s_n\}$ is bounded.
Idea: after a while (eg $n > N$) everything is bounded $s+1$, $s-1$. Take care of finitely many $n \le N$.

Proof   Choose $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ such that for $n > N$, $|s_n - s| < 1$. Let $M = \max\{s_1, \ldots, s_N, s+1\}$, $m = \min\{s_1, \ldots, s_N, s-1\}$.
Let $n \in \mathbb{N}$. If $n \le N$, $m \le s_n \le M$. If $n > N$, $|s - s_n| < 1$,
so $m \le s-1 < s_n < s+1 \le M$. Either way,
$-m \le s_n \le M$. ∎

Proof of Limit Theorem 3)

Idea $|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \le |s_n t_n - s_n t| + |s_n t - st|$ $< \varepsilon$

$$= |s_n||t_n - t| + |t||s_n - s|$$

bounded   Make small

$|s_n| \le M$, $|t_n - t| < \frac{\varepsilon}{2M}$ $|s_n - s| < \frac{\varepsilon}{2(|t|+1)}$

Proof

Let $\varepsilon > 0$. Since $s_n \to s$, there exists $M > 0$ such that $|s_n| < M$ for all $n \in \mathbb{N}$. Choose N_1 such that for $n > N_1$, $|s_n - s| < \frac{\varepsilon}{2(|t|+1)}$

Se that ~~there~~ exists $N_2 \in \mathbb{R}$ such that for $n > N_2$, $|t_n - t| < \frac{\varepsilon}{2M}$.

~~Let~~ Let $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$|s_n t_n - st| \le |s_n||t_n - t| + |t||s_n - s| < M\left(\frac{\varepsilon}{2M}\right) + |t|\frac{\varepsilon}{2(|t|+1)}$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Theorem If $s_n \to s$ such that $s_n \ne 0$ and $s \ne 0$, then $\inf\{|s_n| \mid n \in \mathbb{N}\} > 0$.

Since $|s|/2 > 0$, we can choose $N \in \mathbb{N}$ such that for $n > N$, $|s_n - s| < \frac{|s|}{2}$. Let $m = \min\{|s_1|, \ldots, |s_N|, \frac{|s|}{2}\}$. Then $m > 0$. If $n \in \mathbb{N}$ and $n \le N$, clearly $m \le |s_n|$. If $n > N$, then $|s| - |s_n| \le |s - s_n| < |s|$, so $|s_n| > |s|/2 \ge m$.

So m is a lower bound for $\{|s_n| \mid n \in \mathbb{N}\}$ and $\inf\{|s_n| \mid n \in \mathbb{N}\} \ge m > 0$.

Pf Limit Rule 4

Since $s \neq 0$, $s_n \neq 0$, $m = \inf\{|s_n| \mid n \in \mathbb{N}\} > 0$.

Let $\varepsilon > 0$. Choose N such that for $n > N$, $|s - s_n| < \varepsilon m |s|$.

Then $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| \leq \frac{|s - s_n|}{m|s|} < \frac{\varepsilon m |s|}{m|s|} = \varepsilon$.