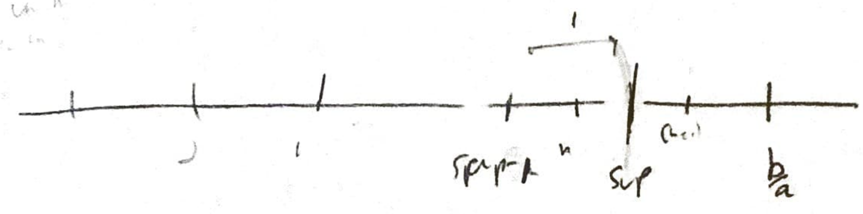


A if and only if B
A iff B
$A \Leftrightarrow B$
~ if A, then B and if B, then A

Proof of Archimedean Principle

Assume the Archimedean Principle fails. Then $\frac{b}{a} \geq n$ for all $n \in \mathbb{N}$, so $\mathbb{N}$ is bounded above by b/a. By the completeness axiom, $r_0 = \sup \mathbb{N}$ exists. Then $r_0 - 1 < r_0$ (and thus $r_0 - 1$ cannot be an upper bound for S.) Thus there exists $n \in \mathbb{N}$ such that $r_0 - 1 < n$. But then $r_0 < n+1$, which contradicts the definition of r_0 as an upper bound for $\mathbb{N}$.

Proof of Denseness of $\mathbb{Q}$ (Sketch) (details Ross 4.7, p.?)

Use Archimedean principle to find $n \in \mathbb{N}$ s.t.
$(b-a)n > 1$
and find $m \in \mathbb{Z}$ such that
$an < m < bn$ (Details?)

Then $a < \frac{m}{n} < b$.

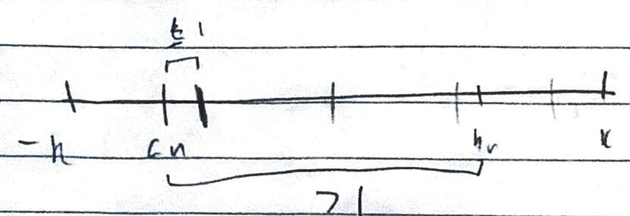
Proof of Denseness of irrationals

Let $r \in \mathbb{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$. Then $a < r + \sqrt{2} < b$. If $r + \sqrt{2} = s \in \mathbb{Q}$, then $\sqrt{2} = s - r \in \mathbb{Q}$, which is false. So $r + \sqrt{2} \notin \mathbb{Q}$.

Details

Choose integer k s.t. $-k < an < bn < k$. Let
$m = \min\{j \in \mathbb{Z} \mid -k \le j \le k,\ j > an\}$, a nonempty finite set.
Then $m > an$, and $m - 1 \le an$. But $bn > an + 1$, so
$m \le an + 1 < bn$.

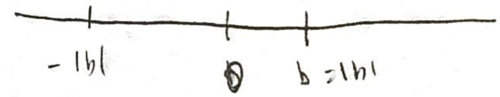
Idea:



count back from k, but stop before an

"smallest integer greater than an"

$-|b| \le b \le |b|$



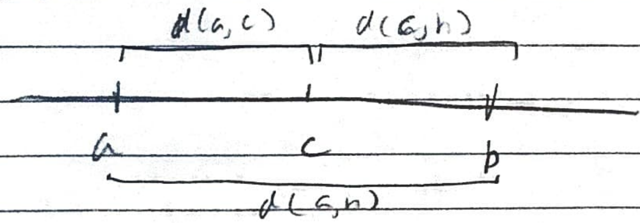
A **metric space** S is a set with a distance function d: for $a, b \in S$, $d(a,b) \in \mathbb{R}$ scalar

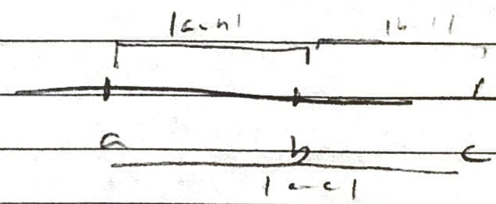
1) $d(a,b) \ge 0$, $d(a,b) = 0 \Rightarrow a = b$
2) $d(a,b) = d(b,a)$
3) $d(a,b) \le d(a,c) + d(c,b)$ for all $a,b,c \in S$.

$\mathbb{R}$ is a metric space with $d(a,b) = |a-b|$.
1) and 2) follow immediately

Proof of 3)
Step 1
$-|a| - |b| \le -|a| + b \le a + b \le a + |b| \le |a| + |b|$
so $a + b \le |a| + |b|$
and $-(a+b) \le |a| + |b|$
$\Rightarrow |a+b| \le |a| + |b|$

Step 2
$d(a,b) = |a-b| = |a-c+c-b| \le |a-c| + |c-b| = d(a,c) + d(c,b)$.





$\boxed{\text{Corollary } |a-b| \ge |a| - |b|}$
$|a| = |a - b + b| \le |a-b| + |b|$

Other metric sc
$\mathbb{R}^2$, $d((x_1, y_1)(x_2, y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$
$\mathbb{R}^2$ $d((x_1,y_1), (x_2,y_2)) = \max\{|x_1-x_2|, |y_1-y_2|\}$
f, g bounded fn $\mathbb{R} \to \mathbb{R}$, $d(f,g) = \sup\{|f(x) - g(x)| \mid x \in \mathbb{R}\}$.

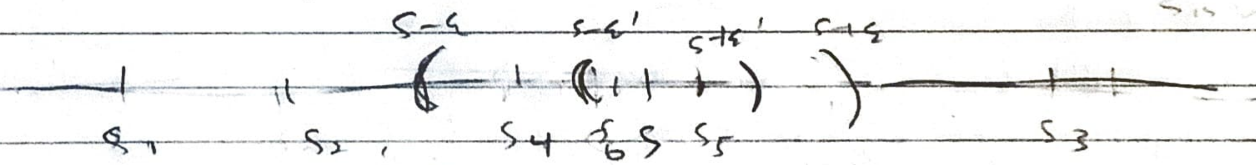
Sequences

A **sequence** is an infinite list of numbers
$(s_m, s_{m+1}, \dots, s_n, \dots) = (s_n)_{n=m}^{\infty}$
started at index, usually 1 = (s_n)
(default)

Note: different from a set, order matters, can have repeats
$(1,2,1,2,\dots)$ different from $(2,1,2,1,\dots)$, both have
"set of values" $\{s_n\} = \{1,2\}$.

Def A sequence (s_n) converges to $s \in \mathbb{R}$ if for all $\varepsilon > 0$, there exists N such that $n > N$ implies $|s_n - s| < \varepsilon$.

(write $s_n \to s$)
$s = \lim s_n$



For $n > 3$, $|s_n - s| < \varepsilon$
for $n > 4$, $|s_n - s| < \varepsilon'$

- Find an N for each ε; N may depend on ε; smaller ε may need a larger N.

- A sequence can have 0 or 1 limit.

Example $s_n = \frac{(-1)^n}{n}$, $s_n \to 0$

Proof Let $\varepsilon > 0$. Choose $N = \frac{1}{\varepsilon}$. If $n > N$,

then $\left| \frac{(-1)^n}{n} - 0 \right| = \frac{1}{n} < \frac{1}{N} = \varepsilon$.

$\varepsilon = 1/4$, $N = 4$