Lecture 2, 8/31/21
Material corresponds to Ross §4.

Completeness

Let $S \subseteq \mathbb{R}$.

$r_0 \in S$ is the **maximum** of S if $r_0 \in S$ and $r \leq r_0$ for all $r \in S$. (write $r_0 = \max S$)
$r_0 \in S$ is the **minimum** of S if $r_0 \in S$ and $r_0 \leq r$ for all $r \in S$. (write $r_0 = \min S$)

$M \in \mathbb{R}$ is an **upper bound** for S if $r \leq M$ for all $r \in S$.
$m \in \mathbb{R}$ is a **lower bound** for S if $m \leq r$ for all $r \in S$.

S is **bounded above** if it has an upper bound.
S is **bounded below** if it has a lower bound.
S is **bounded** if it is bounded above and below.

$r_0 \in \mathbb{R}$ is the **supremum** of S if it is the least upper bound of S. (write $r_0 = \sup S$)
$\iff$ r_0 is an upper bound for S and if r is another upper bound for S then $r_0 \leq r$.
$\iff$ $r_0 = \min\{r \in \mathbb{R} |\ r \text{ is an upper bound for } S\}$.

$r_0 \in S$ is the **infimum** of S if it is the greatest lower bound of S. (write $r_0 = \inf S$)
$\iff$ r_0 is a lower bound for S and if r is another lower bound for S then $r \leq r_0$.
$\iff$ $r_0 = \max\{r \in \mathbb{R} |\ r \text{ is a lower bound for } S\}$.

If $r_0 = \max S$ then $r_0 = \sup S$. If $r_0 = \min S$ then $r_0 = \inf S$.

Completeness Axiom If $S \subset \mathbb{R}$ is nonempty and bounded above then S has a supremum.

Corollary If $S \subset \mathbb{R}$ is nonempty and bounded below then S has an infimum.

Archimedean Principle Let $a, b \in \mathbb{R}$, $a, b > 0$. Then there exists $n \in \mathbb{N}$ such that $na > b$.

Denseness of $\mathbb{Q}$ Let $a, b \in \mathbb{R}$, $a < b$. Then there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Denseness of irrational numbers Let $a, b \in \mathbb{R}$, $a < b$. Then there exists $r \notin \mathbb{Q}$ such that $a < r < b$.

Interval Notation

$[a, b] = \{r \in \mathbb{R} |\ a \leq r \leq b\}$
$(a, b) = \{r \in \mathbb{R} |\ a < r < b\}$
$(a, b] = \{r \in \mathbb{R} |\ a < r \leq b\}$
$[a, b) = \{r \in \mathbb{R} |\ a \leq r < b\}$
$[a, \infty) = \{r \in \mathbb{R} |\ a \leq r\}$
$(a, \infty) = \{r \in \mathbb{R} |\ a < r\}$
$(-\infty, b] = \{r \in \mathbb{R} |\ r \leq b\}$
$(-\infty, b) = \{r \in \mathbb{R} |\ r < b\}$