$r<1$

$2r<1+r$

$r<$

Completeness

$\mathbb{Q}$ has gaps ($\sqrt{2}$) $\mathbb{R}$ should have no gaps

no [illegible] to make this precise, [illegible] completeness rigorous $\Rightarrow$ we state it as an axiom

Let $S \subset \mathbb{R}$.

Definition 1

$r_0 \in S$ is the maximum of S if $r_0 \in S$ and $r_0 \ge r$ for all $r \in S$ (write $r_0 = \max S$)

$r_0 \in S$ is the minimum of S if $r_0 \in S$ and $r_0 \le r$ for all $r \in S$ (write $r_0 = \min S$)

A set can have 0 or 1 max

Examples

$\max[0,1]=1$ $\min[0,1]=0$

$\max\{-4,1,3,17\}=17$ $\min\{-4,1,3,17\}=-4$

Note: a finite set always has a max and min

$[0,\infty)$ has no max, $\min[0,\infty)=0$

$(0,1)$ has no max or min

r is not the min, r is not the max

Notation

"the set of such that is true"

$[a,b] = \{r \in \mathbb{R} \mid a \le r \le b\}$		closed
$[a,b)$	$a \le r < b$	half
$(a,b]$	$a < r \le b$	open
(a,b)	$a < r < b$	open
$[a,\infty)$	$a \le r$	
$(-\infty,a)$	$r < a$	

Definition 2

M is an upper bound for S if $M \ge r$ for all $r \in S$

m is a lower bound for S if $m \le r$ for all $r \in S$

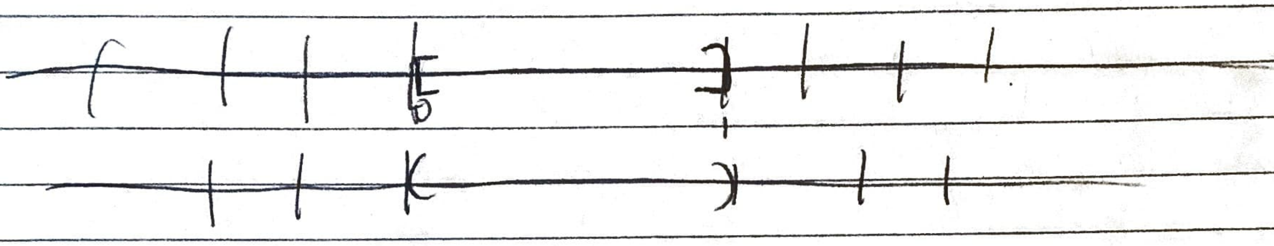
(Same def, but M, m need not be in S)

A set can have 0 or ∞ many upper or lower bounds

Examples

1, any $r > 1$ upper bound $[0,1]$ (max also a upper bound)

0, any $r \le 0$ a lower bound $[0,1]$ (min also a lower bound)



$[0,\infty)$ has no upper bound, 0, negative #'s are lower bounds

Terminology

If S has a lower bound, S is <u>bounded below</u>

upper bound — <u>bounded above</u>

both — <u>bounded</u>

Def 3

$r_0 \in \mathbb{R}$ is the <u>supremum</u> of S if it is the least upper bound (write $r_0 = \sup S$)

(if and only if / equivalent statements) $\Leftrightarrow$, r_0 is an upper bound for S, and if r is an upper bound for S, $r_0 \le r$

$\Leftrightarrow$ $r_0 = \min\{r \in \mathbb{R} \mid r \text{ is an upper bound for } S\}$

$r_0 \in \mathbb{R}$ is the <u>infimum</u> of S if it is the greatest lower bound (write $r_0 = \inf S$)

$\Leftrightarrow$ r_0 is a lower bound for S and if r is a lower bound for S, $r \le r_0$

$\Leftrightarrow$ $r_0 = \max\{r \in \mathbb{R} \mid r \text{ is a lower bound for } S\}$

A set has 1 or 0 sup/inf

$\sup[0,1] = 1$ $\quad \inf[0,1] = 0$

• If $r_0 = \max S$ the $r_0 = \sup S$

Pf:

Suppose $r_0 = \max S$. The r_0 is an upper bound for S and $r_0 \in S$.
If r is an upper bound for S, the $r \ge r_0$. So $r_0 = \sup S$

• If $r_0 = \min S$ the $r_0 = \inf S$

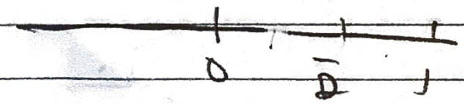
Pf

• $\sup (0,1) = 1$

Pf 1 is an upper bound for $(0,1)$ by definition.
Let r be an upper bound for $(0,1)$. We want to show $r \ge 1$.
If $r < 1$, then $r < \frac{r+1}{2} < 1$ so

$$r < \max\left\{\frac{r+1}{2}, \frac{1}{2}\right\} \in (0,1)$$, and

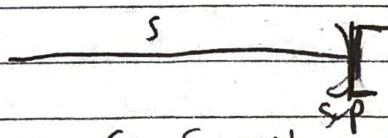
r can't be an upper bound. So $r \ge 1$, and $1 = \sup(0,1)$.



• $\inf (0,1) = 0$.

Completeness Axiom
If $S \subseteq \mathbb{R}$ is non empty and bounded above, it has a supremum

This is "no gaps"



If there were a "gap" we could take $S = \{r \in \mathbb{R} \mid r < \text{gap}\}$

For instance, we will show that $S = \{x \in \mathbb{Q} \mid x^2 \le 2\}$ has $\sup = \sqrt{2}$.
so S has no sup in $\mathbb{Q}$

Corollary If S is bounded below, it has an infimum

Pf (sketch)

If S is bounded below, $-S = \{r \in \mathbb{R} \mid -r \in S\}$ is bounded above (details?)

Then $\sup(-S)$ exists, and $-\sup(-S)$ is the inf of S (details?)

We can use completeness to prove important "intuitive" features of $\mathbb{R}$ "rationals are everywhere"

Archimedean Principle

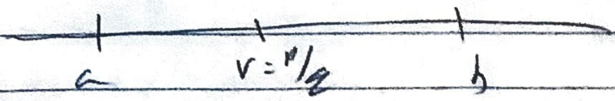
If $a, b > 0$ then exists $n \in \mathbb{N}$ such that $na > b$

Special cases: $a = 1$: for all $b > 0$, there exists $n > b$

$b = 1$: for all $a > 0$, there exists $\frac{1}{n} < a$

Denseness of $\mathbb{Q}$

If $a, b \in \mathbb{R}$, $a < b$, then exists $r \in \mathbb{Q}$ such that $a < r < b$



Denseness of irrationals

If $a, b \in \mathbb{R}$, $a < b$, then exist $r \in \mathbb{R}$, $r \notin \mathbb{Q}$, such that $a < r < b$