Notation $f^{(n)} = (f'''^{\cdots}) = n^{th}$ derivative of f

$f^{(0)} = f$ "n times"

if $f^{(n)}$ exists, we say f has "deriv to order n"

We know that if $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R

$\sum_{n=0}^{\infty} a_n x^n \to f$ (pointwise) on $(-R, R)$

and $\sum_{n=1}^{\infty} n a_n x^{n-1} \to f'$ on $(-R, R)$

$\sum_{n=2}^{\infty} (n-1) n a_n x^{n-2} \to f''$ etc

so $f(0) = a_0$, $1 a_1 = f'(0)$ $f''(0) = 2 \cdot 1 a_2$ $f^{(n)}(0) = n! a_n$

i.e. f has deriv of all orders, and $a_n = \frac{f^{(n)}(0)}{n!}$

What if we start with f?

Def Let I be an open interval containing 0 and $f: I \to \mathbb{R}$ have derivative to order n at 0.

a) $\sum_{K=0}^{n} \frac{f^{(K)}(0)}{K!} x^K$ is the "Taylor poly of degree n for f (centered at 0)"

b) $R_{n+1} = f(x) - \sum_{K=0}^{n} \frac{f^{K}(0)}{K!} x^K$ is the "remainder"

c) If f has deriv of all orders, $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ is the "Taylor series for f" (centered at 0)

Ff $\lim_{n \to \infty} R_n(x) = 0 \iff \sum \frac{f^{K}(0)}{K!} x^K \to f$

Note $\sum \frac{f^{K}(0)}{K!} x^K$ may converge to something other than f

Theorem (Taylor's Theorem) Let I be an open interval containing 0, $f: I \to \mathbb{R}$ diff to order n. Then for $x_0 \neq 0$, $x_0 \in I$, there exists y between 0 and x such that

$$R_n(x_0) = \frac{f^{(n)}(y)}{n!} x_0^n$$

(y depends on x_0 and n)

Proof Let $x_0 \in I$, $x \neq 0$. Let $M := \frac{R(x_0)\, n!}{x_0^n}$. Then

$$f(x_0) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{M x_0^n}{n!}$$

Define $g(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{M x^n}{n!} - f(x)$.

then $g^{(k)}(0) = f^{(k)}(0) - f^{(k)}(0) = 0$ for $k < n$
$g(x_0) = 0$, by choice of M. Thus By The MVT, there
x_1 between 0 and x_0 such that $g'(x_1) = \frac{g(x_0) - g(0)}{x_0 - 0} = 0$

So $g'(0) = g'(x_1)$, there are x_2 between 0 and x_1 so that $g''(x_2) = 0$.
Similarly for $k < n$ there exists x_k such that $g^{(k+1)}(x_k) = 0$.
In particular, $g^{(n)}(x_{n-1}) = M - f^{(n)}(x_{n-1}) = 0$

so $R(x_0) = \frac{f^{(n)}(x_{n-1})}{n!} x_0^n$.

Note: Taylor's Thm requires only n derivs, and is useful itself. In particular, if $f^{(n)}$ is bounded, $\lim_{x \to 0} \frac{R_n(x)}{x^{n-1}} = 0$ (R_n is $o(x^{n-1})$)

In one case, if f has derivs to all orders and $|f^{(k)}(x)| \leq C$ for all $k \in \mathbb{N}$, $x \in I$.
then the series $\sum \frac{f^{(n)}(0)}{n} x^n$ has radius of conv ∞: $\limsup \left|\frac{f^{(n)}(0)}{n!}\right|^{1/n} \leq \lim \frac{C^{1/n}}{(n!)^{1/n}} = 0$

Thm Let I be an open interval cont 0, $f: I \to \mathbb{R}$ diff to all orders,
$|f^{(n)}(x)| \leq C$ for all $n \in \mathbb{N}$, $x \in I$.
$\sum \frac{f^{(n)}(0)}{n!} x^n \to f$ pointwise on I.

Proof

Let $x_0 \in I$. For some $y \in I$,

$$|R_n(x_0)| = \left| \frac{f^{(n)}(y) x_0^n}{n!} \right| \le \frac{C|x_0|^n}{n!} \to 0$$

$\left(\frac{a^n}{n!} \to 0 \right.$ for all $a \in \mathbb{R}$; $\frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n} = \frac{a}{n+1}$, so for $n \ge N$,

$$\left. \frac{a^n}{n!} \le \left(\frac{a}{N+1} \right)^{n-N} a_N \to 0 \right)$$

Example 1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$ $|f^{(n)}(x)| = |\sin(x)|$ or $|\cos(x)| \le 1$

so Taylor series converge to $\sin(x)$ everywhere (without calculators!)

2) $f: [-r, r] \to \mathbb{R}$ $f(x) = e^x$, $|f^{(n)}(x)| = |e^x| \le e^r$.

So $\sum \frac{x^n}{n!} \to e^x$ for $x \in [-r, r]$, and for all $x \in \mathbb{R}$.

3) $f: (-\infty, 1) \to \mathbb{R}$, $f(x) = -\log(1-x)$ ~~not hold for all x~~.

$$f^{(n)}(x) = \frac{(n-1)!}{(1-x)^n} \qquad \frac{f^{(n)}(0)}{n!} = \frac{1}{n}$$

Taylor series $\sum_{n=1}^{\infty} \frac{x^n}{n}$. For $x_0 < 0$, let

$$|R_n(x_0)| = \left| \frac{f^{(n)}(y) x_0^n}{n!} \right| = \left| \frac{x_0^n}{n(1-y)^n} \right| \le \frac{1}{n} \to 0$$

We saw radius of conv was $[-1, 1)$. Now we can $\sum \frac{(-1)^n}{n} \to \log(2)$.

You will show $f(x) = e^{-\frac{1}{x^2}}$ for $x \ne 0$, $f(0) = 0$ has $f^{(n)}(0) = 0$ for all n.

So Taylor series is $0 \ldots$