Def Given $a_k \in \mathbb{R}$ for all $k \in \mathbb{N}$, let $g_n(x) = a_k x^n$.
The series $\sum g_n = \sum a_n x^n$ is called a <u>power series</u>.

<u>Question</u> On what set S does $\sum a_n x^n$ converge uniformly?

<u>Def</u> (Radius of Convergence)
Let $\sum a_n x^n$ be a power series.
If $(|a_n|^{1/n})$ is unbounded, define $R = 0$.
Otherwise, let $\beta = \limsup |a_n|^{1/n}$.
If $\beta = 0$, define $R = \infty$.
If $\beta > 0$, then $R = 1/\beta$.

R is called the "radius of convergence" of $\sum a_n x^n$

<u>Recall</u> If $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$ converges, then $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim |a_n|^{1/n}$.

<u>Thm</u> Let $\sum a_n x^n$ be a power series with radius of convergence R.
a) $\sum a_n x^n$ converges for $|x| < R$ (i.e. on ... conv pointwise)
b) $\sum a_n x^n$ does not conv for $|x| > R$

(Note: all power series conv at $x = 0$)

In other words, if $g_n(x) = a_n x^n$, $g_n : (-R, R) \to \mathbb{R}$, then $\sum g_n$ conv pointwise.

<u>Proof</u> <u>Case 1</u> $R \in (0, \infty)$. Let ... Apply root test to $\sum a_n x^n$.
$\limsup |a_n x^n|^{1/n} = \limsup |a_n|^{1/n} |x| = |x| \limsup |a_n|^{1/n} = |x| \beta$.
So $\sum a_n x^n$ conv if $|x|\beta = \frac{|x|}{R} < 1$ and diverges if $\frac{|x|}{R} > 1$.

<u>Case 2</u> $R = \infty$. $\beta = 0$ always, so $\sum a_n x^n$ conv for all x.

<u>Case 3</u> $R = 0$. The $(|a_n|^{1/n} |x|)$ is unbounded (if $|x| > 0$), so $\sum a_n x^n$ does not conv for $x \neq 0$.

Note: anything can happen at $x = \pm R$.

Def Let $\sum a_n x^n$ be a power series. The interval of conv is the set of $x \in \mathbb{R}$ such that $\sum a_n x^n$ converges.

Note If $\sum a_n x^n$ has radius of conv R, $(-R,R)$ is contained in the interval of conv, and no x with $|x|>R$ is. Thus the interval of conv is $(-R,R)$, $[-R,R]$, $(-R,R]$ or $[-R,R)$.

Examples 1) $a_n = 1$; $\sum x^n$

$\beta = \limsup 1^{1/n} = \limsup 1 = 1$, so $R = 1$.

When $x=1$, $\sum 1$ diverges; when $x=-1$, $\sum (-1)^n$ does not conv.

The interval of conv is $(-1,1)$ (geometric series)

2) $\sum \frac{x^n}{n}$; $a_n = 1/n$. $|a_n|^{1/n} = \frac{1}{n^{1/n}} \to 1$, so $R=1$ $\left(\frac{n+1}{n} = 1 + \frac{1}{n} \to 1\right)$

When $x=1$, $\sum \frac{1}{n}$ does not conv; $x=-1$ $\sum \frac{(-1)^n}{n}$ does conv (alternating)

The interval of conv is $[-1,1)$.

3) $\sum \frac{(-x)^n}{n}$; $a_n = \frac{(-1)^n}{n}$, interval of conv is $(-1,1]$

4) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n}$ $\quad a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{(-1)^{n/2}}{n/2} & \text{if } n \text{ is even} \end{cases}$

$-x^2 + \frac{x^4}{2} - \frac{x^6}{3}$

$\left|\frac{2}{n}\right|^{1/n} \to 1$ $\left(\frac{n+1}{2} \cdot \frac{2}{n} = 1 + \frac{1}{n} \to 1\right)$

$x = \pm 1$ $\sum \frac{(-1)^n}{n}$ conv, interval of conv is $[-1,1]$.

If I is the interval of conv of $\sum a_n x^n$, let $g_n: I \to \mathbb{R}$, $g_n(x) = a_n x^n$. $\sum g_n$ conv pointwise. Write "$\sum a_n x^n$ conv pointwise on I"

(or $R=\infty$)

Theorem Let $\sum a_n x^n$ be a power series with radius of convergence $R>0$. If $0<R_1<R$ (or $0<R_1$ if $R=\infty$) then $\sum a_n x^n$ conv unif on $[-R_1,R_1]$.

Proof

So $|a_n x^n| \le |a_n| R_1^n$ for all $x \in [-R_1, R_1]$, by the M test $\sum a_n x^n$ conv uniformly if $\sum |a_n| R_1^n$ converges.

By the root test, $\limsup (|a_n| R_1^n)^{1/n} = \limsup |a_n|^{1/n} R_1 = R_1 \limsup |a_n|^{1/n} = R_1 \beta = \begin{cases} 0 \\ \frac{R_1}{R} \end{cases} < 1$

So $\sum |a_n| R_1^n$ converge.

Corollary $\sum a_n x^n$ conv to a continuous fn on $(-R, R)$.

Proof Let $x_0 \in (-R,R)$. Let $x_0 < R_1 < R$. $\sum a_n x^n$ conv uniformly on $[-R_1, R_1]$, so the limit is continuous on $[-R_1,R_1]$ so thus cont at x_0.

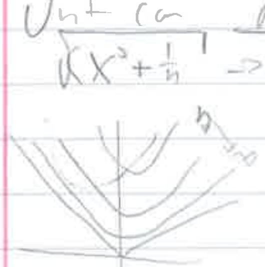
Differentiation of fns

Thm Let $f_n : [a,b] \to \mathbb{R}$ be cont and diff on (a,b) for all $n \in \mathbb{N}$. If $f_n' \to g$ unif. and $(f_n(x_0))$ conv for some $x_0 \in (a,b)$, then there is $f:[a,b] \to \mathbb{R}$ such that $f_n \to f$ and $f' = g$.

Need "$(f_n(x_0))$ conv" is clearly necessary: Ex, $f_n(x) = n$, $f_n' = 0$, so $f_n' \to 0$, but (f_n) does not conv.
(if you find the derivs, you need to find the fn up to a con)

Unif conv does not eq differentiability, or limit of deriv's deriv

$\sqrt{x^2+\frac{1}{n}} \to |x|$ uniformly



$\frac{x^n}{n} \to 0$ unif