Lecture 20, 11/9/21
Material corresponds to Ross §34.

Properties of Integrals

Theorem Let $f, g : [a, b] \to \mathbb{R}$ be functions. If f is integrable and $f(x) = g(x)$ for all $x \in [a, b]\backslash S$, where S is finite, then g is integrable and $\int_a^b f = \int_a^b g$.

Definition $f : [a, b] \to \mathbb{R}$ is **piecewise continuous** if there exists a partition

$$P = \{a = t_0 < t_1 < ... < t_n = b\}$$

of $[a, b]$ such that $f : (t_{k-1}, t_k) \to \mathbb{R}$ is uniformly continuous for all $k = 1, ..., n$.

Theorem A piecewise continuous function is integrable.

Fundamental Theorem of Calculus

Definition We say $f : (a, b) \to \mathbb{R}$ is **integrable on** $[a, b]$ if any extension of $f : [a, b] \to \mathbb{R}$ is integrable. $\int_a^b f$ does not depend on the extension.

Fundamental Theorem of Calculus I If $f : [a, b] \to \mathbb{R}$ is continuous, and differentiable on (a, b), and $f' : (a, b) \to \mathbb{R}$ is integrable on $[a, b]$, then

$$\int_a^b f' = f(b) - f(a).$$

Definition If $a > b$, $\int_a^b f = -\int_b^a f$.

Fundamental Theorm of Calculus II Let $f : [a, b] \to \mathbb{R}$ be integrable. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f.$$

Then F is continuous. If f is continuous at $x_0 \in (a, b)$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Theorem (Integration by Parts) Let $u, v : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b) such that $u', v' : (a, b) \to \mathbb{R}$ are integrable on $[a, b]$. Then

$$\int_a^b uv' + \int_a^b u'v = u(b)v(b) - u(a)v(a).$$

Theorem (Change of Variables) Let J be an open interval $u : J \to \mathbb{R}$ a differentiable function such that $u' : J \to \mathbb{R}$ is continuous. Let I be an open interval such that $u(J) \subset I$ and let $f : I \to \mathbb{R}$ be continuous. For all $a, b \in J$

$$\int_a^b (f \circ u)u' = \int_{u(a)}^{u(b)} f.$$