<u>Theorem</u> Let $f, g: [a,b] \to \mathbb{R}$ be functions. If f is integrable, and $f(x) = g(x)$ for $x \in [a,b] \setminus S$, where S is finite, then g is integrable and $\int_a^b f = \int_a^b g$

<u>Proof</u>
$(g-f)(x) = 0$ for $x \in [a,b] \setminus S$. By HW, $g-f$ is int. and $\int_a^b (g-f) = 0$, so $g = (g-f) + f$ is int and
$\int_a^b g = \int_a^b (g-f) + \int_a^b f = \int_a^b f$.

<u>Def</u> $f:[a,b] \to \mathbb{R}$ is piecewise continuous if there is a partition $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ such that $f: (t_{k-1}, t_k) \to \mathbb{R}$ is <u>unif</u> contin for each k.

<u>Thm</u> A piecewise continuous function is integrable

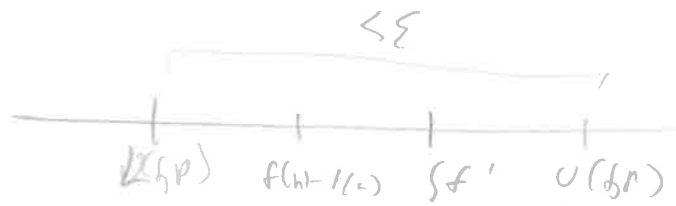
<u>Proof</u> $f:(t_{k-1}, t_k) \to \mathbb{R}$ can be extd to a contin function on $[t_{k-1}, t_k]$, differs from f on $[t_{k-1}, t_k]$ by at most 2 points. So $f: [t_{k-1}, t_k]$ is integrable.

<u>Def</u> $f:(a,b) \to \mathbb{R}$ is called integrable on $[a,b]$ if an extension to $[a,b]$ is integrable

<u>Theorem</u> (Fundamental Theorem of Calculus I)
If $f:[a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b), and $f':(a,b) \to \mathbb{R}$ is int on $[a,b]$, then
$\int_a^b f' = f(b) - f(a)$.

$\{a = t_0 < \dots < t_k < \dots < t_n\}$

<u>Proof</u> Let $\varepsilon > 0$. Since f' is integrable, there exists a partition P of $[a,b]$ such that $U(f', P) - L(f', P) < \varepsilon$. By the MVT, there exists $x_k \in (t_{k-1}, t_k)$ such that $f'(x_k)(t_k - t_{k-1}) = f(t_k) - f(t_{k-1})$. So
$f(b) - f(a) = \sum_{k=1}^{n} f(t_k) - f(t_{k-1}) = \sum f'(x_k)(t_k - t_{n-1})$.
$f(t_n) - f(t_{n-1}) + f(t_{n-1}) - f(t_{n-\dots}$



Since $m(f',[t_{k-1},t_k]) \le f'(x_k) \le M(f',[t_{k-1},t_k])$
$L(f',P) \le f(b)-f(a) \le U(f',P)$. P, defn.
$L(f',P) \le \int_a^b f' \le U(f',P)$, so

$$\left| \int_a^b f' - (f(b)-f(a)) \right| < \varepsilon .$$

Since this is true for all $\varepsilon > 0$, $\int_a^b f' = f(b) - f(a)$.

Convention: If $a > b$, $\int_a^b f := -\int_b^a f$

Theorem (Fund. Thm. of Calc. II)
Let $f : [a,b] \to \mathbb{R}$ be integrable. Define $F : [a,b] \to \mathbb{R}$,
$F(x) = \int_a^x f$ $\quad \left(F(x) = \int_a^x f(t)\,dt \right)$

Then F is continuous. If f is continuous at $x_0 \in (a,b)$,
F is diff. at x_0, and $F'(x_0) = f(x_0)$.

Proof Let $\varepsilon > 0$.
Choose M such that $|f(x)| \le M$ for $x \in [a,b]$. Set $\delta = \frac{\varepsilon}{M}$.
If $x, y \in [a,b]$, $x < y$ and $y - x < \delta$, then
$$|F(y) - F(x)| = \left| \int_a^y f - \int_a^x f \right| = \left| \int_x^y f \right| \le \int_x^y |f| \le \int_x^y M = M(y-x) < \varepsilon .$$

So F is continuous.
Now assume f is continuous at $x_0 \in (a,b)$. Note for $x \ne x_0$,
$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f \quad \text{and} \quad f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x f(x_0).$$
so $$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x f - f(x_0)$$

Let $\varepsilon > 0$. Choose δ such that for $x \in (a,b)$ and $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \frac{\varepsilon}{2}$.

For such $x \neq x_0$,

$$\left|\frac{F(x)-f(x_0)}{x-x_0} - f(x_0)\right| = \left|\frac{1}{x-x_0}\right| \left|\int_{x_0}^{x} f - f(x_0)\right| \leq \frac{1}{|x-x_0|}\int_{x_0}^{x} |f-f(x_0)|$$

$$\leq \frac{1}{|x-x_0|}\frac{\varepsilon}{2}|x-x_0| = \frac{\varepsilon}{2} < \varepsilon.$$

So $\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$. ∎

Application of Fund. Thm. of Calc. to derivative rules for integral rules.

Thm $u, v : (a,b) \to \mathbb{R}$ continuous, diff on (a,b), u', v' integrable on $[a,b]$, then

$$\int_a^b uv' + \int_a^b u'v = u(b)v(b) - u(a)v(a)$$

Proof $g = uv$ is diff by the product rule so $g' = u'v + vu'$ is integ. Since u', v', u, g are +I(V). So $\int g' = g'(b) - g'(a)$.

Thm Let J be an open interval, $u : J \to \mathbb{R}$ diff, $u' : J \to \mathbb{R}$ continuous. Let $I \subset \mathbb{R}$ be an open interval, $u(J) \subset I$, $f : I \to \mathbb{R}$ continuous. For all $a, b \in J$

$$\int_a^b (f\circ u)u' = \int_{u(a)}^{u(b)} f$$

Proof
Fix $c \in I$, define $F : I \to \mathbb{R}$, $F(y) = \int_c^y f$. Since f is cont, F is diff and $F' = f$. So $g = F\circ u$ is diff, and $g'(x) = F'(u(x))u'(x) = f(u(x))u'(x)$. $f\circ u$, u' are continuous, so g' is continuous and integrable. Thus

$$\int_a^b (f\circ u)u' = \int_a^b g' = g(b) - g(a) = F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} f.$$

∎

$$\int_{a}^{b} x e^{x^2}\,dx \qquad u = x^2 \quad du = 2x\,dx$$

$$= \int_{a^2}^{b^2} x e^{u} \frac{du}{2x} = \frac{1}{2}\int_{a^2}^{b^2} e^{u}\,du = \frac{1}{2}\left(e^{b^2} - e^{a^2}\right).$$

$u : (a,b) \to \mathbb{R}$, $\underset{u' = 2x \text{ cont}}{u(x) = x^2}$ $\quad f : (a^2, b^2) \to \mathbb{R}$, $\underset{\text{cont}}{f(y) = e^{y}}$.

$$\int_{a}^{b} x e^{x^2} = \frac{1}{2}\int_{a}^{b} (2x) e^{x^2} = \frac{1}{2}\int_{a}^{b} u'(f \circ u) = \frac{1}{2}\int_{u(a)}^{u(b)} f$$