Goal: Prove things about numbers, sequences and functions

Some are familiar from calculus, other fundamental to further applications

Intermediate value thm, mean value thm
L'Hopital,
series tests, Maclaurin
Fundamental thm calc

sets in Metric spaces
uniform continuity
Sequences of functions
uniform convergence

Start with a set of axioms (algebraic rules, statements) taken as true without proof

Rigor	Utility
Everything we prove depends on those assumptions	Know what tools we have to prove things

	operations	Axioms
$\mathbb{N} = \{1, 2, 3, \dots\}$	$+, \times, \leq$	Induction Axiom
$\cap$		
$\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$	$+, -, \times, \leq$	
$\cap$		
$\mathbb{Q} \; \{\frac{p}{q} \mid p, q \in \mathbb{N}, q \neq 0\}$	$+, -, \times, \div, \leq$	"ordered field axioms"
$\cap$		
$\mathbb{R} = ?$ real numbers		Completeness

Notation

Sets have elements: "a is in S" "a is an element of S"

$S = \{a, b, c\}$ $\quad a \in S$

(repeats, order don't mean anything: $\{1,1\} = \{1\}$ $\quad \{a,b\} = \{b,a\}$)

$S \subseteq T$: if $x \in S$ then $x \in T$ / $x \in S \overset{\text{"implies"}}{\Rightarrow} x \in T$

$S \subset T$: $S \subseteq T$ but $S \neq T$ (there exist $x \in T$ such that $x \notin S$).

Ordered Field Axioms (True for $\mathbb{Q}$ and $\mathbb{R}$)

Given $a, b \in \mathbb{Q}$ ($\mathbb{R}$), $a+b \in \mathbb{Q}$ ($\mathbb{R}$) and $a \cdot b \in \mathbb{Q}$ ($\mathbb{R}$)

+ Addition is commutative, $a+b=b+a$; associative, $(a+b)+c=a+(b+c)$; has 0, $a+0=a$; has inverse, $a+(-a)=0$

• Multiplication is commutative, $ab=ba$; associative, $(ab)c=a(bc)$; distributive, $a(b+c)=ab+ac$; has 1, $a\cdot 1=a$; has inverse (if $a \neq 0$), $a(a^{-1})=1$, $a \neq 0$

$\leq$ For $a,b \in \mathbb{Q}$ ($\mathbb{R}$) either $a \leq b$ or $b \leq a$
If $a \leq b$ and $b \leq a$ then $a=b$
If $a \leq b$ and $b \leq c$ then $a \leq c$
If $a \leq b$ then $a+c \leq b+c$
If $a \leq b$ and $0 \leq c$ then $ac \leq bc$

We can now prove all "common sense" facts

~~Example~~

1) $(-a)(-b) = ab$ (Special case $a=-1$: $(-1)b = (1)(-b) = -b$)

Proof

Using distributive, $(-a)(-b) + a(-b) = (-a+a)(-b)$
$= 0(-b) = 0.$

So $(-a)(-b)$ is $\boxed{\text{the}}$ inverse of $a(-b)$.
Since $a(-b) + ab = a(-b+b) = a(0) = 0$
so $a(-b)$ is the inverse of ab.
Thus $(-a)(-b)$ is $\boxed{\text{the inverse of the inverse}}$ of ab

$\mathbb{N}$ is a special subset of $\mathbb{Q}$ / $\mathbb{R}$ satisfying the
Induction Axiom

If $S \subset \mathbb{N}$, $1 \in S$, and $n \in S \Rightarrow n+1 \in S$, then $S = \mathbb{N}$

1 2 3 4 → "all in a line"

This is how proof by induction works:

① Create a list of statements $P_1, P_2, P_3, \dots$
② Show P_1 is true
③ Show $P_n \Rightarrow P_{n+1}$
④ conclude P_n is true for all $n \in \mathbb{N}$

Why? Let $S = \{ n \in \mathbb{N} \mid P_n \text{ is true} \} \subset \mathbb{N}$)
②: $1 \in S$ ③: $n \in S \Rightarrow n+1 \in S$ Induction Axiom $\Rightarrow$ ④

Example Prove $(1+2+\dots+n)^2 = 1^3+2^3+\dots+n^3$ for all $n \in \mathbb{N}$.
(& this is P_n).

② when $n=1$ $1^2 = 1 = 1^3$ (P_1 is true)

③ Now Assume $(1+\dots+n)^2 = 1^3+\dots+n^3$. Then

$$(1+\dots+n+n+1)^2 = (1+\dots+n)^2 + 2(1+\dots+n)(n+1) + (n+1)^2$$
$$= 1^3+\dots+n^3 + \frac{2(n)(n+1)^2}{2} + (n+1)^2$$

Using the formula $(1^2+\dots+m) = m(m+1)/2$
we reach
$(1+\dots n+n+1)^2 = 1^3+\dots+n^3+(n+1)^3$, and the conclusion by induction. $\boxtimes$

What is missing from $\mathbb{Q}$?

↪ lots of solutions to algebraic equations, like $\sqrt{2}$

Theorem (Rational zeros theorem)

Let $r = \frac{p}{q} \in \mathbb{Q}$. be a solution to polynomial in x

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

for $a_n, \dots, a_1, a_0 \in \mathbb{Z}$. Then $r = \frac{p}{q}$ such that

q divides a_n and p divides a_0.