Let

A bounded function

Recall $f:[a,b]\to\mathbb{R}$ is integrable if and only if for every $\varepsilon>0$ there exists a partition P of $[a,b]$ such that

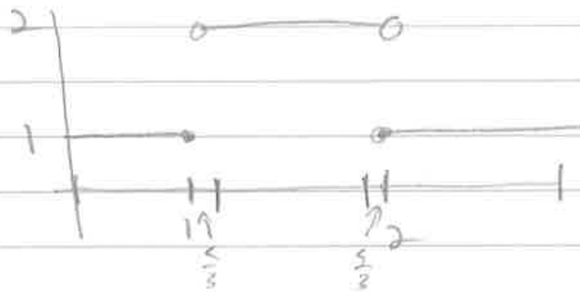
$$U(f,P)-L(f,P)<\varepsilon.$$

(Often we use $U(f,P)\ge\int_a^b f\ge L(f,P)$)

Example $f:[0,3]\to\mathbb{R}$ $\quad f(x)=1$ for $x\in[0,1]\cup[2,3]$

$f(x)=2$ for $x\in(1,2)$

Let $\varepsilon>0$. Let $P=\{0,1,1+\frac{1}{3}\varepsilon,2-\frac{1}{3}\varepsilon,2,3\}$



$$U(f,P)=1(1-0)+2\left(1+\tfrac{1}{3}\varepsilon-1\right)+2\left(2-\tfrac{1}{3}\varepsilon-\left(1+\tfrac{1}{3}\varepsilon\right)\right)+2\left(2-\left(2-\tfrac{1}{3}\varepsilon\right)\right)+(3-2)(1)$$

$$=1+\frac{2}{3}\varepsilon+2\left(1-\frac{2}{3}\varepsilon\right)+\frac{2}{3}\varepsilon+1=4$$

$$L(f,P)=1(1-0)+1\left(\tfrac{1}{3}\varepsilon\right)+2\left(1-\tfrac{2}{3}\varepsilon\right)+1\left(\tfrac{1}{3}\varepsilon\right)+1$$

$$=4-\frac{2}{3}\varepsilon$$

so $U(f,P)-L(f,P)=\frac{2}{3}\varepsilon<\varepsilon$, and f is integrable. Then

$4\ge\int_a^b f\ge 4-\frac{2}{3}\varepsilon$ for all ε, so $\int_a^b f=4$

Example 2 $f:[0,1]\to\mathbb{R}$ $\quad f(x)=0$ if $x\in\mathbb{R}\setminus\mathbb{Q}$

$\qquad\qquad\qquad\qquad\quad f(x)=1$ if $x\in\mathbb{Q}$

For any $x<y$, $x,y\in[0,1]$, $M(f,[x,y])=1$ $\;m(f,[x,y])=0$ by density of $\mathbb{Q}$, $\mathbb{R}\setminus\mathbb{Q}$. So $U(f,P)=1$, $L(f,P)=0$ for all P, so $U(f)=1$ & $L(f)=0$.

Properties of Integrals

Let $f,g:[a,b]$ be bounded

Theorem 1

a) If f is monotonic, then f is integrable

b) If f is continuous, then f is integrable

Let $f,g:[a,b]\to\mathbb{R}$ be integrable

Theorem 2 (Linearity) $\quad c\in\mathbb{R}$

a) cf is integrable and $\int_a^b cf = c\int_a^b f$

b) $f+g$ is integrable and $\int_a^b f+g = \int_a^b f + \int_a^b g$

Theorem 3 order

a) If $f(x)\le g(x)$ for all $x\in[a,b]$, $\int_a^b f\le\int_a^b g$

b) $|f|$ is integrable and $\left|\int_a^b f\right| \le \int_a^b |f|$

c) If f is continuous and $f(x)\ge 0$ for all $x\in[a,b]$, and $\int_a^b f(x)=0$

then $f(x)=0$ for all $x\in[a,b]$.

Theorem 4 Suppose $f:[a,c]\to\mathbb{R}$ is integrable, $f:[c,b]\to\mathbb{R}$ is integrable. Then $f:[a,b]\to\mathbb{R}$ is integrable & $\int_a^b f=\int_a^c f+\int_c^b f$

Proof Ideas (+ Picture)

Thm 4

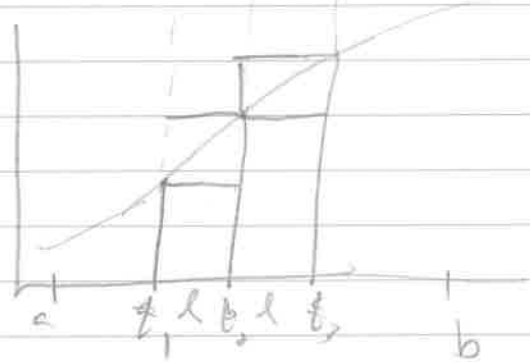
a) $U(f,P) - L(f,P) =$

$(f(t_2) - f(t_1))\ell + (f(t_3) - f(t_2))\ell$

$= (f(t_3) - f(t_1))\ell + \dots$

$= \dots (f(b) - f(a))\ell$

s.t. $\ell = \frac{\varepsilon}{f(b) - f(a)}$

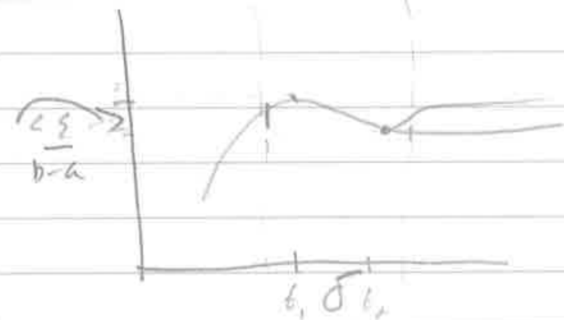


b) f is unif. cont. $\frac{\varepsilon}{b-a} \rightsquigarrow \delta$

$M(f,[t_1,t_2]) - m(f,[t_1,t_2]) < \frac{\varepsilon}{b-a}$

$U(f,P) - L(f,P) < \sum \frac{\varepsilon}{b-a}(t_k - t_{k-1})$

$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon$



Thm 2 a

c) If $c > 0$, $U(cf,P) = cU(f,P)$

also, $U(-f) = -L(f)$, etc.

b) $M(f(x)+g(x), S) \le M(f(x),S) + M(g(x),S)$

$m(f(x)+g(x), S) \ge m(f(x),S) + m(g(x),S)$

$L(f) + L(g) \le L(f+g) \le U(f+g) \le U(f) + U(g)$