Lecture 18, 11/2/21
Material corresponds to Ross §32.

The Integral

Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

Definitions

- Let $S \subseteq [a,b]$. Define $M(f,S) = \sup\{f(x) | x \in S\}$ and $m(f,S) = \inf\{f(x) | x \in S\}$.
- A **partition** of $[a,b]$ is a finite subset $P \subset [a,b]$ containing a and b, labeled
$$P = \{a = t_0 < t_1 < ... < t_{n-1} < t_n = b\}.$$
- Given a partition P of $[a,b]$ the **upper and lower Darboux sums** of f are
$$U(f,p) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1})$$
$$L(f,p) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1})$$
- the **upper and lower Darboux integrals** of f are
$$U(f) = \inf\{U(f,P) | \text{ all partitions } P \text{ of [a,b] }\}$$
$$L(f) = \sup\{U(f,P) | \text{ all partitions } P \text{ of [a,b] }\}$$
- f is **integrable** if $U(f) = L(f)$. In that case the integral is defined as
$$\int_a^b f = U(f) = L(f).$$

Lemmas

- If P, Q are partitions of $[a,b]$ such that $P \subset Q$ then
$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$
- If P, Q are partitions of $[a,b]$ then $L(f,P) \leq U(f,Q)$.
- $L(f) \leq U(f)$.

Theorem f is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that
$$U(f,P) - L(f,P) < \epsilon.$$