Let $f : [a,b] \to \mathbb{R}$ be bounded

Notation For $S \subseteq [a,b]$

$$M(f,S) = \sup\{f(x) \mid x \in S\}$$
$$m(f,S) = \inf\{f(x) \mid x \in S\}$$

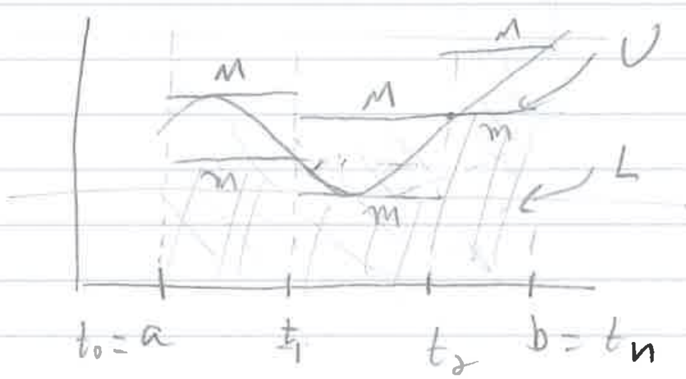
A partition of $[a,b]$ is a finite subset $P \subset [a,b]$ containing $a, b$

Notation: $P = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$

Def "Darboux Sums"

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}) \quad \leq M(f,[a,b]) \underbrace{\sum_{k=1}^{n} (t_k - t_{k-1})}_{b-a}$$

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$$



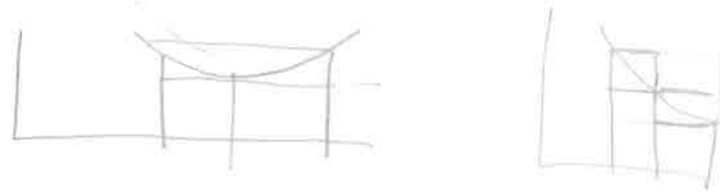

$$m(f,[a,b]) \leq m(f,[t_{k-1},t_k]) \leq M(f,[t_{k-1},t_k]) \leq M(f,[a,b])$$

so

$$m(f,[a,b])(b-a) \leq L(f,P) \leq U(f,P) \leq M(f,[a,b])(b-a)$$

$$U(f) = \inf\{U(f,P) \mid \text{all partn } P \text{ of } [a,b]\}$$
$$L(f) = \sup\{L(f,P) \mid \text{all partn } P \text{ of } [a,b]\}$$

Def $f$ is integrable on $[a,b]$ if $U(f) = L(f)$. Then $\int_a^b f = L(f) = U(f)$



**Lemma** If $P \subseteq Q$ then
$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$
✓

**Proof**

Assume $P = \{a = t_0 < t_1 < \dots < t_n = b\}$
$Q' = \{a = t_0 < \dots < t_{k-1} < u < t_k < \dots < t_n = b\}$

That is, $Q' = P \cup \{u\}$, $t_{k-1} < u < t_k$


$$L(f,Q) - L(f,P) = m(f,[t_{k-1},u])(u - t_{k-1}) + m(f,[u,t_k])(t_k - u) - m(f,[t_{k-1},t_k])(t_k - t_{k-1})$$
$$\ge m(f,[t_{k-1},t_k])(u - t_{k-1} + t_k - u - t_k + t_{k-1})$$
$$= 0$$

The case for $U$ is similar. We keep adding points to get $Q$.

**Lemma** If $P, Q$ are partitions of $[a,b]$, then
$$L(f,P) \le U(f,Q)$$

**Proof** $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$ so
$$L(f,P) \le L(f,P\cup Q) \le U(f,P\cup Q) \le U(f,Q).$$

**Theorem** $L(f) \le U(f)$

**Proof** Let $P$ be a partition of $[a,b]$. For all partitions $Q$, $L(f,P) \le U(f,Q)$. So $L(f,P)$ is a lower bound for $\{U(f,Q) \mid Q \text{ partition of } [a,b]\}$
so $L(f,P) \le U(f)$.
Since this is true for all partitions $P$, $U(f)$ is an upper bound for $\{L(f,P) \mid P \text{ partition of } [a,b]\}$
so $L(f) \le U(f)$.

**Theorem** $f$ is integrable if and only if for each $\varepsilon > 0$ there exists a partition $P$ of $[a,b]$ such that
$$U(f,P) - L(f,P) < \varepsilon.$$

**Proof**

**Step 1** Assume $f$ is integrable. Let $\varepsilon > 0$.

Since $L(f) - \frac{\varepsilon}{2} < L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$

there exists a partition $P_1$ such that $L(f) - \frac{\varepsilon}{2} < L(f,P_1)$.

Similarly, there exists a partition $P_2$ such that $U(f) + \frac{\varepsilon}{2} > U(f,P_2)$

Let $P = P_1 \cup P_2$

Then
$$\begin{aligned} U(f,P_1) - L(f,P) &\le U(f,P_1) - L(f,P_2) \\ &< U(f) + \tfrac{\varepsilon}{2} - L(f) + \tfrac{\varepsilon}{2} \\ &= U(f) - L(f) + \varepsilon \\ &= \varepsilon \end{aligned}$$

**Step 2** Assume for each $\varepsilon$ there exists a partition $P$ such that
$U(f,P) - L(f,P) < \varepsilon$. Then
$$U(f) \le U(f,P) < L(f,P) + \varepsilon \le L(f) + \varepsilon.$$

Since this is true for all $\varepsilon$, $U(f) \le L(f)$. But $L(f) \le U(f)$ always,
so $L(f) = U(f)$.