Lecture 17, 10/28/21
Material corresponds to Ross §30.

L'Hospital's Theorem

Definition Let (s_n) be a sequence in $\mathbb{R}$.

1. We say $s_n \to \infty$ if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that if $n > N$ then $s_n > M$
2. We say $s_n \to -\infty$ if for all $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that if $n > N$ then $s_n < M$.

Definition Let $E \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$ such that there is a sequence in E converging to a. Let $f : E \to \mathbb{R}$ be a function.

1. We say $\lim_{x\to a} f(x) = \infty$ if for every sequence (s_n) in E converging to a, $f(s_n) \to \infty$.
2. We say $\lim_{x\to a} f(x) = -\infty$ if for every sequence (s_n) in E converging to a, $f(s_n) \to -\infty$.

Theorem Let $E \subseteq \mathbb{R}$ and let $a \in \mathbb{R}$ such that there is a sequence in E converging to a. Let $f : E \to \mathbb{R}$ be a function.

1. $\lim_{x\to a} f(x) = \infty$ if and only if for all $M \in \mathbb{R}$ there exists $\delta > 0$ such that $x \in E$ and $|x - a| < \delta$ implies $f(x) > M$.
2. $\lim_{x\to a} f(x) = -\infty$ if and only if for all $M \in \mathbb{R}$ there exists $\delta > 0$ such that $x \in E$ and $|x - a| < \delta$ implies $f(x) < M$.

Theorem

1. Let (s_n) be a sequence in $\mathbb{R}$ such that $s_n \neq 0$ for all $n \in \mathbb{N}$. If $s_n \to \pm\infty$, $\frac{1}{s_n} \to 0$
2. Let $E \subset \mathbb{R}$ and $a \in \mathbb{R}$ such that there exists a sequence in E converging to a. Let $f : E \to \mathbb{R}$ be a function such that $f(x) \neq 0$ for all $x \in E$. If $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} \frac{1}{f(x)} = 0$.

Theorem (L'Hospital's Rule) Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions, and assume $g'(x) \neq 0$ for all $x \in (a, b)$. If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \qquad \text{or} \qquad \lim_{x\to a} \frac{f'(x)}{g'(x)} = \pm\infty$$

and either

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \text{or} \qquad \lim_{x\to a} |g(x)| = \infty$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$