$s_n \to \infty$ if for all $M \in \mathbb{R}$, there is $N \in \mathbb{R}$ such that if $n > N$, $s_n > M$.
$(-\infty)$ $(< M)$

$f: E \to \mathbb{R}$, $a \in \mathbb{R}$ a limit of a seq in E

$\lim_{x \to a} f(x) = \infty$ (or $-\infty$) if

a) For all $M \in \mathbb{R}$, there exists $\delta > 0$ such that if $x \in E$, $|x-a| < \delta$, then $f(x) > M$ (or $< M$)

b) For all (s_n) in E, if $s_n \to a$ then $f(s_n) \to \infty$.

<u>Thm</u> a) If $x_n \to \pm\infty$, then $\frac{1}{x_n} \to 0$
b) If $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a} \frac{1}{f(x)} = 0$.

<u>Proof</u> Let $\varepsilon > 0$. Then exists $N \in \mathbb{R}$ such that for $n > N$, $|x_n| > \frac{1}{\varepsilon}$. Then for $n > N$, $\left|\frac{1}{x_n}\right| < \varepsilon$. b) follows immediately.

<u>L'Hospital's Rule</u>

Let $f, g : (a,b) \to \mathbb{R}$ be differentiable, $g'(x) \neq 0$ for $x \in (a,b)$
If $\lim_{x \to a} = L \in \mathbb{R}$, or $\pm\infty$ and

<u>Case I</u>: $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or <u>Case II</u>: $\lim_{x \to a} g(x) = \pm\infty$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

<u>Proof</u>
If there are $x, y \in (a,b)$ such that $g'(x) < 0$ and $g'(y) > 0$ then there would be $x_0 \in (x,y)$ such that $g'(x_0) = 0$ (IVT for derivs).
So either $g'(x) > 0$ for all $x \in (a,b)$, or $g'(x) < 0$ for all $x \in (a,b)$.
Replace g w/ $-g$ if necessary. Assume $\boxed{g'(x) < 0}$ for all $x \in (a,b)$.
So g is decreasing. If $g(x_0) = 0$, $x_0 \in (a,b)$, then for $x < x_0$, $g(x) \neq 0$. Change b to x_0, we can assume $\boxed{g(x) \neq 0}$ for all $x \in (a,b)$.

Assume $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ or $-\infty$

Let $L < K < M$.
(if $-\infty$, any $K < M$)

There exists $\delta > 0$ such that if $x \in (a, a+\delta)$ then
$\frac{f'(x)}{g'(x)} < K$.

Let $y \in (a, a+\delta)$. For all $x \in (a, y)$ there exists $x_0 \in (x, y)$ (GMVT)
such that $\frac{f(x)-f(y)}{\underbrace{g(x)-g(y)}_{>0}} = \frac{f'(x_0)}{\underbrace{g'(x_0)}_{<0}} < K$.

Case I

Then $\lim_{x\to a} \frac{f(x)-f(y)}{g(x)-g(y)} = \frac{\lim_{x\to a} f(x) - f(y)}{\lim_{x\to a} g(x) - g(y)} = \frac{f(y)}{g(y)} \le K < M$

$\boxed{\text{So if } y \in (a, a+\delta),\ \frac{f(y)}{g(y)} < M}$

Case II Since $\lim g(x)$ and $g' < 0$, we must have $\lim_{x\to a} g(x) = \infty$ 
as $g(x) > 0$

For $y \in (a, a+\delta)$

$$f(x) < K(g(x)-g(y)) + f(y)$$

$$\frac{f(x)}{g(x)} < K\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)}$$

$$\lim_{x\to a} K\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} = K\left(1 - g(y)\lim_{x\to a}\frac{1}{g(x)}\right) + f(y)\lim_{x\to a}\frac{1}{g(x)}$$
$$= K$$

So there exists δ_2 such that if $x \in (a, a+\delta_2)$

$$\frac{f(x)}{g(x)} < K\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} < M$$

$\boxed{\text{So for } x \in (a, a+\delta_2),\ \frac{f(x)}{g(x)} < M}$

In either case, we can find δ such that for $x \in (a, a+\delta)$
$\frac{f(x)}{g(x)} < M$. If $\lim_{x \to a} \frac{f'(x)}{g'(x)} = -\infty$, this is true for any M, so

$$\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty.$$

let $\varepsilon > 0$.
If $L \in \mathbb{R}$, choose $M = L + \varepsilon$. Then $x \in (a, a+\delta)$, $\frac{f(x)}{g(x)} < L + \varepsilon$

And can make a similar argument that, $\exists\, x \in (a, a+\delta')$, $\frac{f(x)}{g(x)} > L - \varepsilon$.

Thus $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

<u>Recall</u>

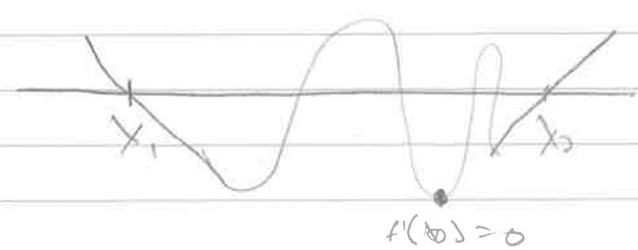
<u>Generalized Mean Value Theorem</u> (GMVT)

Let $f, g : [a,b] \to \mathbb{R}$ be continuous and diff on (a,b). Then exists $x \in (a,b)$ such that

$$f'(x)\,(g(b) - g(a)) = g'(x)\,(f(b) - f(a)).$$

<u>IVT for derivatives</u> Let I be an open interval.
Let $f : I \to \mathbb{R}$ be diff. If c is between $f'(x_1)$ and $f'(x_2)$, $x_1, x_2 \in I$
then exists $x_0 \in I$ such that $f'(x_0) = c$

<u>Pf</u> $g = f - cx$ $\quad$ $f'(x_1) < c < f'(x_2)$
$g'(x_1) < 0 < g'(x_2)$



examples: $\lim_{a \to 0} \frac{\sin(x)}{x} = \lim_{a \to 0} \frac{\cos(x)}{1} = 1$

$\lim_{a \to \infty} \frac{e^x}{x} = \lim_{a \to \infty} e^x = \infty$

Indeed $\lim_{a \to \infty} \frac{e^x}{x^n} \underset{L'H}{=} \lim_{a \to \infty} \frac{e^x}{nx^{n-1}} \underset{L'H}{=} \cdots \underset{L'H}{=} \lim_{a \to \infty} \frac{e^x}{n!} = \infty$

By continuity of log,

$\log\left(\lim_{x \to \infty} x^{1/x}\right) = \lim_{x \to \infty} \log(x^{1/x}) = \lim_{x \to \infty} \frac{\log(x)}{x} = \lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x \to \infty} x^{1/x} = e^0 = 1.$