Lecture 15, 10/26/21 Material corresponds to Ross §29.

Mean Value Theorem

Theorem Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable at $x_0 \in I$. If f assumes its maximum at x_0 ($f(x_0) \ge f(x)$ for all $x \in I$) or f assumes its minimum at x_0 ($f(x_0) \ge f(x)$ for all $x \in I$) then $f'(x_0) = 0$.

Rolle's Theorem Let f : [a, b] be a continuous function which is differentiable on (a, b). If f(a) = f(b) then there exists $x \in (a, b)$ such that f'(x) = 0.

Generalized Mean Value Theorem (GMVT) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions which are differentiable on (a, b). Then there exists $x \in (a, b)$ such that

$$f'(x)(g(b) - g(a)) = g'(x)(f(b) - f(a)).$$

Corollary (Meant Value Theorem, MVT) Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then there exists $x \in (a, b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Corollary Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function. If $f' : I \to \mathbb{R}$ is bounded then f is uniformly continuous.

Corollary Let $I \subseteq \mathbb{R}$ be an open interval and let $f: I \to \mathbb{R}$ be a differentiable function.

- 1. If f'(x) > 0 for all $x \in I$ then f(b) > f(a) for all $a, b \in I$, a < b.
- 2. If $f'(x) \ge 0$ for all $x \in I$ then $f(b) \ge f(a)$ for all $a, b \in I$, a < b.
- 3. If f'(x) < 0 for all $x \in I$ then f(b) < f(a) for all $a, b \in I$, a < b.
- 4. If $f'(x) \leq 0$ for all $x \in I$ then $f(b) \leq f(a)$ for all $a, b \in I$, a < b.
- 5. If f'(x) = 0 for all $x \in I$ then f(b) = f(a) for all $a, b \in I$.

Theorem (Intermediate Value Theorem for Derivatives) Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be a differentiable function. If $x_1, x_2 \in I$, $x_1 < x_2$, and c lies between $f'(x_1)$ and $f'(x_2)$ then there exists $x \in (x_1, x_2)$ such that f'(x) = c.