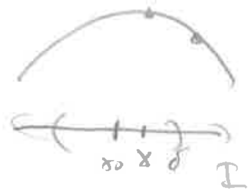

Mean Value Theorem

Thm Let I be an open interval, $x_0 \in I$, $f: I \to \mathbb{R}$ diff at x_0.
If f attains its maximum at x_0 ($f(x_0) \ge f(x)$ all $x \in I$)
or f attains its minimum at x_0 ($f(x_0) \le f(x)$ $x \in I$)
Then $f'(x_0) = 0$

Proof
Assume f is diff at x_0 and $f(x_0) \ge f(x)$ for all $x \in I$.
Assume towards a contradiction $f'(x_0) > 0$.
Let $\varepsilon = f'(x_0)$. There exists δ such that for $x \in I$, $|x - x_0| < \delta$,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < f'(x_0), \text{ so } \frac{f(x) - f(x_0)}{x - x_0} > 0$$

This is a contradiction for $x \in I$, $x > x_0$, $|x - x_0| < \delta$.

Similarly, if $f'(x_0) < 0$, there exists δ such that for $x \in I$, $|x - x_0| < \delta$, $\frac{f(x) - f(x_0)}{x - x_0} < 0$, a contradiction for $x \in I$, $x < x_0$, $|x - x_0| < \delta$.

Rolle's Theorem

Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b).
If $f(a) = f(b)$, there exists $x \in (a,b)$ such that $f'(x) = 0$.

Proof Since f is continuous on $[a,b]$, there exists $x_0, y_0 \in [a,b]$ such that $f(x_0) \le f(x) \le f(y_0)$ for all $x \in I$.
If x_0, y_0 are both endpoints, $f(x_0) = f(y_0)$, so f is constant and $f' = 0$ on (a,b).

Otherwise, x_0 or $y_0 \in (a,b)$ and $f'(x_0)$ or $f'(y_0) = 0$ by previous theorem.

(Generalized) Mean Value Thm

Let $f, g : [a,b] \to \mathbb{R}$ be continuous, and diff on (a,b). Then exists $x \in (a,b)$ such that

$$f'(x)\,(g(b)-g(a)) = g'(x)\,(f(b)-f(a)).$$

Proof

Let $h(x) = f(x)\,(g(b)-g(a)) - g(x)\,(f(b)-f(a))$

$$h(a) = f(a)g(b) - g(a)f(b)$$

$$h(b) = -f(b)g(a) + g(b)f(a) = h(a)$$

So there exists $x \in (a,b)$ such that $h'(x) = 0$.

Corr Mean val th

Let $f : [a,b]$ be contin, and diff on (a,b), then exists $x \in (a,b)$ such that $f'(x) = \dfrac{f(b)-f(a)}{b-a}$

Proof G'MVT with $g(x) = x$.

Cor Let $f : (a,b) \to \mathbb{R}$ be diff such that $f'(x) = 0$ for all $x \in (a,b)$. Then f is constant.

Proof Assume f is not constant. Then exists $x_1 < x_2 \in (a,b)$ such that $f(x_1) \neq f(x_2)$. Then there exists $x \in (x_1, x_2)$ such that $f'(x) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$.

Thm Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable. If f' is bounded, then f is uniformly continuous.

Proof
Suppose $|f'(x)| \le M$ for all x, $M>0$. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{M}$. Let $x<y \in I$, $|x-y|<\delta$. Then there exists $z \in (x,y)$ such that $\frac{f(y)-f(x)}{y-x} = f'(z)$, so

$$|f(y)-f(x)| = |y-x||f'(z)| < \delta M = \varepsilon.$$

Theorem Let I be an open interval and $f: I \to \mathbb{R}$ diff'l.
If $f'(x) > 0$ for all $x \in I$, $f(y) > f(x)$ for all $y>x$, $x,y \in I$ (increasing)

$\ge \Rightarrow \ge$ (nondecreasing)

$< \Rightarrow <$

$\le \Rightarrow \le$

Proof $\exists\, x_0 \in (x,y)$ such that $f'(x_0) = \frac{f(y)-f(x)}{y-x} > 0$, etc.

Thm (IVT for derivatives)
Let I be an open interval, and let $f: I \to \mathbb{R}$ be diff'l on I. If $x_1 < x_2$, $x_1, x_2 \in I$ and c lies between $f'(x_1)$ and $f'(x_2)$, then there exists $x \in (x_1, x_2)$ such that $f'(x) = c$

Note! This would be true by IVT if $f': I \to \mathbb{R}$ were continuous. But that is not always the case; see HW. Many IVT prop still holds. (i.e. $f'(I)$ is an interval)

Proof Assume $f'(x_1) < c < f'(x_2)$. Let $g(x) = f(x) - cx$.
Then $g'(x_1) < 0 < g'(x_2)$. $g' = f' - c$
The minimum x_0 of $[x_1, x_2]$ such that $g(x_0) \le g(x)$ for all $x \in [x_1, x_2]$.
Since $g'(x_1) < 0$, there is $\delta > 0$ such that for $y \in (x_1, x_1+\delta)$,
$\frac{g(y)-g(x_1)}{y-x_1} < 0$, so $g(y) < g(x_1)$, and $x_0 \ne x_1$. Similarly, $x_0 \ne x_2$.
so $x_0 \in (x_1, x_2)$ and $g'(x_0) = 0$, so $f'(x_0) = c$.