E is disconnected if there exist open $A, B \in S$ such that

1) $E \subset A \cup B$
2) $A \cap B \cap E = \emptyset$
3) $A \cap E \neq \emptyset$
4) $B \cap E \neq \emptyset$

Thm 1 $E \subseteq \mathbb{R}$ is connected if and only if E is an interval

Step 1 If E is not an interval, E is disconnected ✓

Step 2 Let E be an interval. Assume, towards contradiction, that E is disconnected.
Let A, B be open sets in $\mathbb{R}$ satisfying 1–4.
Let $a \in A \cap E$, $b \in B \cap E$; assume $a < b$; then $[a,b] \subseteq E$.
Let $t = \sup [a,b] \cap A$. Then $t \in [a,b] \subseteq E$.
There exists a seq (x_n) in $[a,b] \cap A$ such that $x_n \to t$.
So $x_n \in E \cap A$, $x_n \notin B$. Since B is open, $\mathbb{R} \setminus B$ is closed, so $t \in \mathbb{R} \setminus B$.
In particular, $t < b$. Let $y_n = t + \frac{b-t}{n}$.

Then $t < y_n < b$ and $y_n \to t$. Since $y_n \in [a,b]$, $y_n > t$, $y_n \notin A$. Since A is open, $t \notin A$. So $t \notin A \cup B$, contradiction ①.

Thm 2 If $E \subseteq S$ is connected, $f : S \to S^*$ continuous, then $f(E)$ is connected.

Proof Assume $f(E)$ is not connected. We show E is not connected.
Let A, B be open sets in S^* such that
$f(E) \subset A \cup B$ $\quad f(E) \cap A \cap B = \emptyset$ $\quad f(E) \cap A \neq \emptyset$ $\quad f(E) \cap B \neq \emptyset$
By continuity, $f^{-1}(A)$ and $f^{-1}(B)$ are open sets in S.
$E \subset f^{-1}(A) \cup f^{-1}(B)$ $\quad E \cap f^{-1}(A) \cap f^{-1}(B) = \emptyset$ $\quad E \cap f^{-1}(A) \neq \emptyset$ $\quad E \cap f^{-1}(B) \neq \emptyset$ ☒

Def Let I be an open interval, $a \in I$, $f : I \to \mathbb{R}$.
Then $\frac{f(x) - f(a)}{x - a} : I \setminus \{a\} \to \mathbb{R}$. f is *differentiable at a* if

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. If the limit exists, it is called $f'(a)$.

Recall $\lim_{x \to a}$ is allowed on $I \setminus \{a\}$ since there is a sequence in $I \setminus \{a\}$ converging to a
(This is often the kind of limit we care about)

$f: I \to \mathbb{R}$ is differentiable if it is differentiable at every point of I. In that case, $f': I \to \mathbb{R}$ is a function.

Example $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

$$\frac{f(x)-f(a)}{x-a} = \frac{x^2-a^2}{x-a} = \frac{(x-a)(x+a)}{x-a} = x+a \quad : \mathbb{R} \setminus \{a\} \to \mathbb{R}$$

$$\lim_{x \to a} x+a = a+a = 2a \qquad f'(a) = 2a$$

$f': \mathbb{R} \to \mathbb{R}$, $f'(x) = 2X$.

Theorem Let I be an open interval, $f: I \to \mathbb{R}$
If f is differentiable at $a \in I$, then f is continuous at $a \in I$.

Idea: if $\frac{f(x)-f(a)}{x-a} \approx f'(a)$, $|f(x)-f(a)|$ "<" $|x-a||f'(a)|$
$< \delta |f'(a)|$
$< \varepsilon$

Proof
Let $\varepsilon > 0$. There exists δ_1 such that if $x \in I \setminus \{a\}$, $|x-a| < \delta_1$,

$$\left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < \frac{\varepsilon}{2}.$$

Let $\delta_2 = \min\left\{ \frac{\varepsilon}{2|f'(a)|}, 1, \delta_1 \right\}$ (if $f'(a)=0$, just $1, \delta_1$). Then for $x \in I$, $|x-a| < \delta_2$

$$\left| \frac{f(x)-f(a)}{x-a} \right| \le \frac{\varepsilon}{2} + |f'(a)|$$

$$|f(x)-f(a)| \le \frac{\varepsilon}{2}|x-a| + |f'(a)||x-a|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Careful with δ?

Derivative Theorems

1) $(cf)'(a) = c(f'(a))$

2) $(f+g)'(a) = f'(a) + g'(a)$

3) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

4) $(1/x)'(a) = -\frac{1}{a^2}$

Proof Apply lmt thms to

1) $\frac{cf(x) - cf(a)}{x-a} = c\left(\frac{f(x)-f(a)}{x-a}\right)$

2) $\frac{f(x)+g(x) - (f(a)+g(a))}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}$

3) $\frac{f(x)g(x) - f(a)g(a)}{x-a} = \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a}$

$= f(x)\left(\frac{g(x)-g(a)}{x-a}\right) + g(a)\left(\frac{f(x)-f(a)}{x-a}\right)$

4) $\frac{\frac{1}{x}-\frac{1}{a}}{x-a} = \frac{1}{x-a}\frac{a-x}{xa} = -\frac{1}{xa}$

Chain Rule

given interval I_1, $a \in I_1$, $f: I_1 \to \mathbb{R}$ diffable at a
given interval I_2, $f(I_1) \subset I_2$, $g: I_2 \to \mathbb{R}$ diffable at $f(a)$
then $g \circ f: I_1 \to \mathbb{R}$ is diffable at a and
$(g \circ f)'(a) = g'(f(a))f'(a)$.

Idea:

$\frac{g(f(x)) - g(f(a))}{f(x)-f(a)} \frac{f(x)-f(a)}{x-a} = \frac{g(f(x)) - g(f(a))}{x-a}$

$\downarrow$ $\downarrow$

$g'(f(x_0))$ [illegible] $f'(a)$ but $f(x)-f(a)$ maybe 0 for $x \neq a$.

(Ross 28.4)

Proof

Case 1 There is an interval $I \subset I_1$ such $f(x) \neq f(a)$ $\forall x \in I \setminus \{a\}$.

Let (x_n) be a seq in $I_1 \setminus \{a\}$, $x_n \to a$. There is $N \in \mathbb{N}$ such that $x_n \in I \setminus \{a\}$, and then $f(x_n) \neq f(a)$, for $n > N$.

Since f is continuous at a, $y_n = f(x_n) \to f(a)$. Then

$$\frac{(g\circ f)(x_n) - g\circ f(a)}{x_n - a} = \underset{\substack{\downarrow \\ g'(f(a))}}{\left(\frac{g(y_n)-g(f(a))}{y_n - f(a)}\right)}\underset{\substack{\downarrow \\ f'(a)}}{\left(\frac{f(x_n)-f(a)}{x_n-a}\right)}$$

So $\lim_{y\to f(a)} \frac{g(y)-g(f(a))}{y-f(a)} = g'(f(a))$,

$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$

Combining results, we have $(g\circ f)'(a) = g'(f(a))f'(a)$.

Case 2 There exists a seq z_n in $I_1 \setminus \{a\}$ such that $f(z_n) = f(a)$. Then

$$\frac{f(z_n)-f(a)}{z_n-a} = 0 \to f'(a), \text{ so } f'(a) = 0.$$

We just need to check $(g\circ f)'(a) = 0$ in this case.

Idea: for y near $f(a)$ $\frac{g(f(y))-g(f(a))}{y-f(a)} \approx g'(f(a))$

so $|g(y)) - g(f(a))| \le C|y - f(a)|$

For any seq $x_n \to a$, large n,

$$\frac{|g(f(x_n)) - g(f(a))|}{|x_n - a|} \le C\frac{|f(x_n)-f(a)|}{|x_n-a|}$$