Let S be a metric space with distance d

Recall 1) An open cover of $E \subseteq S$ is a collection $C = \{U_\alpha \mid \alpha \in A\}$ (any set) of open sets $U_\alpha \subseteq S$ such that each element of E is contained in some U_α

$$(E \subseteq \bigcup_{\alpha \in A} U_\alpha)$$

2) A finite subcover of C is a finite subset $\{U_{\alpha_1}, \dots, U_{\alpha_n}\} \subseteq C$ such that every element of E is in some $U_{\alpha_1}, \dots, U_{\alpha_n}$

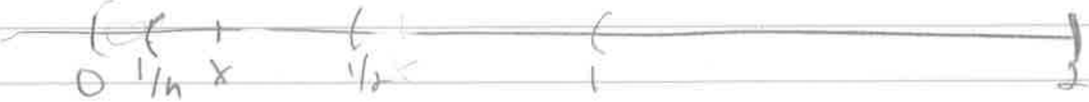
$$(E \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n})$$

Def E is compact if every open cover contains a finite subcover.

(Note: every set has a finite cover ($\{S\}$), but E may have another cover with no finite subcover)

Example 1 $C = \{(1,2), (\frac{1}{2},2), (\frac{1}{3},2), \dots, (\frac{1}{n},2), \dots\}$ is an open cover of $(0,2)$

$x \in (0,2)$, $0 < x < 2$ then $n \in \mathbb{N}$ such that $\frac{1}{n} < x < 2$, $x \in (\frac{1}{n}, 2)$.



But any finite subset of C is of the form $\{(\frac{1}{n_1},2), \dots, (\frac{1}{n_k},2)\}$ $n_1 < n_2 < \dots < n_k$ and $\frac{1}{n_k} \in (0,2)$ is in none of these sets. So C has no finite subcover.

Example 2 C is not an open cover of $[0,2]$, $0,2 \notin (\frac{1}{n},2)$. $C \cup \{(-1,1), (1,3)\}$ is an open cover of $[0,2]$, with finite subcover $\{(-1,1), (\frac{1}{2},2), (1,3)\}$.

Theorem: K is compact if and only if it is sequentially compact

General metric space	$\mathbb{R}^k$
Compact $\overset{\text{proved}}{\Longleftrightarrow}$ Sequentially Compact	Compact $\Leftrightarrow$ SC
$\Downarrow$ $\not\Uparrow$	$\Downarrow$ $\Uparrow$
closed + bounded	Closed + bdd

Can use this def to prove previous theorems

<u>Thm</u> $E \subseteq S$ compact, $f: E \to S'$ contin, then
1) $f(E)$ is compact 2) f is uniformly contin

<u>Proof idea</u>
1) $\{U_\alpha\}$ open cover of $f(E)$, $\{f^{-1}(U_\alpha)\}$ open cover of $E \xrightarrow{\text{compact}}$ finite sub $\{f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})\}$ of $f^{-1}(E)$ $\Rightarrow$ finite $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ of $f(E)$

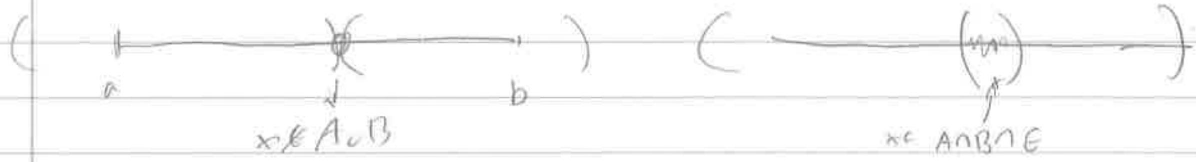
2) let $\varepsilon > 0$. For each $x \in E$, there δ_x such that $d(s,x) < \delta \Rightarrow d^*(f(s), f(x)) < \varepsilon$.
$U_x = \{s \in S \mid d(s,x) < \delta_x/2\}$ is a cover, a finite subcover gives $\delta_{x_1}, \ldots, \delta_{x_n}$, the min ($> 0$) is δ.

<u>Connectedness</u>

A set $E \subseteq S$ is <u>disconnected</u> if there are open sets $A, B \subseteq S$ such that
1) $E \subseteq A \cup B$
2) $A \cap B \cap E = \emptyset$
3) $A \cap E \neq \emptyset$
4) $B \cap E \neq \emptyset$

A set is <u>connected</u> if it is not disconnected.

Idea! Can't do this with an interval $E = [a,b]$ or (a,b)



Theorem 1 $E \subseteq \mathbb{R}$ is connected if and only if it is an interval

Let S be a metric space with metric d; let S^* be a metric space with metric d^*

Theorem 2 If $E \subseteq S$ is connected, $f: S \to S^*$ is continuous, then $f(E)$ is connected.

Corollary If $E \subseteq S$ is connected, $f: E \to \mathbb{R}$ is continuous, $f(E)$ is an interval; i.e. f has the intermediate value property

($a, b \in E$, y between $f(a)$ and $f(b)$, there is $x \in E$ such that $f(x) = y$)

Proof Thm 1

Step 1 Assume E is not an interval; prove E is not connected.

Since E is not an interval, there are $a, b \in E$, and $x \notin E$ such that $a < x < b$.

Let $A = (-\infty, x)$ and $B = (x, \infty)$.

1) $E \subseteq \mathbb{R} \setminus \{x\} = A \cup B$
2) $A \cap B = \emptyset$
3) $a \in A \cap E$
4) $b \in B \cap E$

So E is not connected

Step 2 Assume E is an interval. Assume, towards a contradiction, that E is not connected. Let A, B be as in the defn of this result.

so $[a,b] \subseteq E$

Let $a \in A \cap E$, $b \in B \cap E$. Let $t = \sup [a,b] \cap A$. Then $t \in [a,b] \subseteq E$

there is a seq (x_n) in $[a,b] \cap A$ such that $x_n \to t$. Since $x_n \in A \cap E$, $x_n \notin B$. Since B is open, $\mathbb{R} \setminus B$ is closed, and $x_n \to t$ implies $t \in \mathbb{R} \setminus B$, and implies $t < b$.

Let $y_n = t + \frac{b-t}{n}$. Then $t < y_n < b$ and $y_n \to b$. Since $y_n > t$, $y_n \notin A \cap [a,b]$, so $y_n \notin A$. But A is open, so $y_n \to t$ implies $t \in \mathbb{R} \setminus A$. So $t \notin A \cup B$, a contradiction.