Lecture 13, 10/7/21
Material corresponds to Ross §13 and §21.

Continuity in Metric Spaces

Let S be a metric space with distance function d.
Let S^* be a metric space with distance function d^*.

Definition Let $E \subseteq S$. A function $f : S \to S^*$ **is continuous at** $x_0 \in E$ if

- for all sequences (x_n) in E, $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.
- OR EQUIVALENTLY, If for all $\epsilon >$ there exists $\delta > 0$ such that $x \in E$, $d(x, x_0) < \delta$ implies $d^*(f(x), f(x_0)) < \epsilon$.

Definition Let $E \subseteq S$. A function $f : E \to^*$ is **uniformly continuous** if for all $\epsilon > 0$ there exists $\delta > 0$ such that $x, y \in E$, $d(x, y) < \delta$ implies $d^*(f(x), f(y)) < \epsilon$.

Definition Let $E \subseteq S$, $f : E \to S^*$, and $E^* \subseteq S^*$. Then the **inverse image** of E^* is

$$f^{-1}(E^*) = \{x \in E | f(x) \in E^*\}.$$

Theorem $f : S \to S^*$ is continuous if and only if for ever open set $U \subseteq S^*$, $f^{-1}(U)$ is an open set in S.

Theorem Let $E \subseteq S$ be sequentially compact and let $f : E \to S^*$ be continuous. Then:

1. $f(E)$ is sequentially compact.
2. f is uniformly continuous.

Theorem Let $E \subseteq S$ be sequentially compact and let $f : E \to \mathbb{R}$ be continuous. Then

1. $f(E)$ is bounded.
2. There exists $x_0, y_0 \in E$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in E$.

Definition

1. An **open cover** of $E \subseteq S$ is a collection $C = \{U_\alpha |\ \alpha \in A\}$ of open sets $U_\alpha \subset S$ such that ever element of E is contained in some U_α, that is, $E \subseteq \bigcup_{\alpha \in A} U_\alpha$.
2. A **finite subcover** of C is a finite subset $\{U_{\alpha_1}, ..., U_{\alpha_k}\} \subset C$ such that $E \subseteq U_{\alpha_1} \cup ... \cup U_{\alpha_k}$.

Definition A set $E \subset S$ is **compact** if every open cover of E contains a finite subcover.