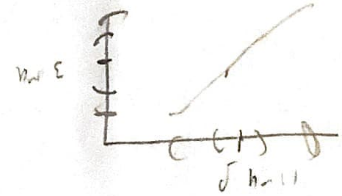


Continuity in Metric Spaces

Let S be a metric space with distance function d
Let S^* be a metric space with distance function d^*

Definition Let $E \subseteq S$. A function $f: E \to S^*$ is continuous at $x_0 \in E$ if

1) for all sequences (x_n) in E, $x_n \to x_0$ implies $f(x_n) \to f(x_0)$

OR

2) for all $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in E$, $d(x, y) < \delta$ implies $d^*(f(x), f(y)) < \varepsilon$.

Defn $f: E \to S^*$ is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in E$, $d(x,y) < \delta$ implies $d^*(f(x), f(y)) < \varepsilon$

Example If $E \subseteq \mathbb{R}$, $f: E \to \mathbb{R}^k$, then f is (unif) continuous if and only if $f(x) = (f_1(x), \ldots, f_k(x))$ and each $f_i: E \to \mathbb{R}$ is (unif) continuous

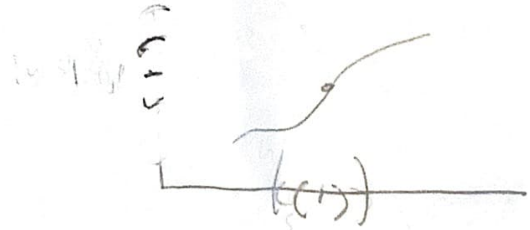
Def $E \subseteq S$, $f: E \to S^*$, $E^* \subseteq S^*$, $f^{-1}(E^*) = \{x \in E \mid f(x) \in E^*\}$. inverse image

Theorem $f: S \to S^*$ is continuous if and only if for every open set $U \subseteq S^*$, $f^{-1}(U)$ is open.

Proof Step 1 Assume $f: S \to S^*$ is continuous.
Let $U \subseteq S^*$ be open. Let $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$ and there exists $\varepsilon > 0$ such that $\{y \in S^* \mid d^*(y, f(x_0)) < \varepsilon\} \subseteq U$.
By continuity there exists $\delta > 0$ such that $x \in S$, $d(x, x_0) < \delta$ implies $d^*(f(x), f(x_0)) < \varepsilon$, which in turn implies $f(x) \in U$.
Thus $\{x \in S \mid d(x, x_0) < \delta\} \subseteq f^{-1}(U)$. This proves $f^{-1}(U)$ is open.

Step 2 Assume $f^{-1}(U)$ is open for every open set $U \subseteq S^*$.
Let $x_0 \in S$ and let $\varepsilon > 0$. The set $U = \{y \in S^* \mid d^*(y, f(x_0)) < \varepsilon\}$ is open. Thus $f^{-1}(U)$ is open; note $x_0 \in f^{-1}(U)$. Thus there exists $\delta > 0$ such that $\{x \in S \mid d(x, x_0) < \delta\} \subseteq f^{-1}(U)$.

HW4 Sol



Thus, if $x \in S$, $d(x, x_0) < \delta$, then $f(x) \in \mathcal{U}$ and $d^*(f(x), f(x_0)) < \varepsilon$.

Theorem Let $E \subseteq S$ be sequentially compact, let $f: E \to S^*$ be continuous. Then

1) $f(E)$ is sequentially compact
2) f is uniformly continuous.

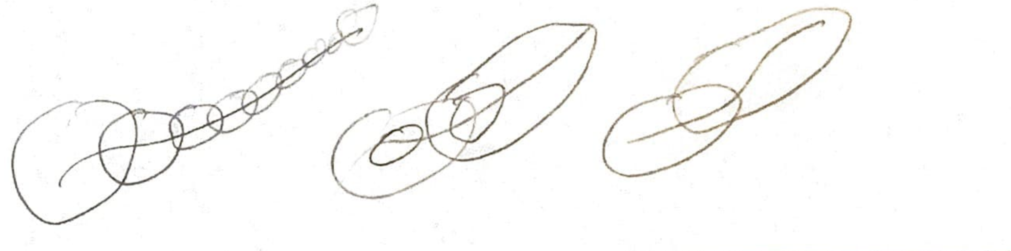
Proof
1) Let (y_n) be a sequence in $f(E)$. Then $y_n = f(x_n)$ for some $x_n \in E$. Since E is SC, there is a subsequence x_{n_k} converging to $x_0 \in E$. By continuity, $f(x_{n_k}) = y_{n_k} \to f(x_0) \in f(E)$. Thus $f(E)$ is SC.

2) Repeat proof that if $f: [a,b] \to \mathbb{R}$ is continuous, it is uniformly continuous. (SC, A)

Theorem Let $E \subseteq S$ be sequentially compact. Let $f: E \to \mathbb{R}$ be continuous. Then
1) $f(E)$ is bounded
2) There exist $x_0, y_0 \in E$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in E$.

(i.e. $f(x_0) = \min f(E)$, $f(y_0) = \max f(E)$, f "achieves its max/min")

Proof $f(E) \subseteq \mathbb{R}$ is S.C. so $f(E)$ is closed and bounded. Let $M = \sup f(E)$. There is a seq (x_n) in $f(E)$ converging to M but E is closed, so $M \in f(E)$. Thus there exists $y_0 \in E$ such that $f(y_0) = M = \max f(E)$. $f(x_0) = \min f(E)$ is similar.



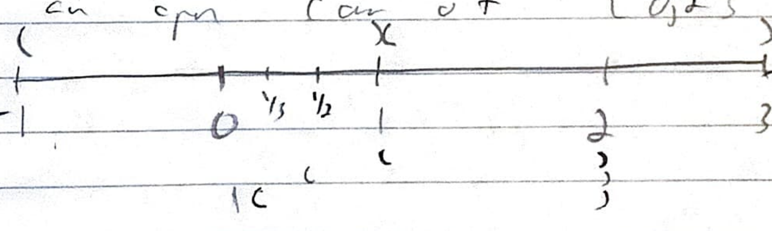
Def 1) An open cover of $E \subseteq S$ is a collection of open sets U_α, $C = \{U_\alpha \mid \alpha \in A\}$ such that $E \subseteq \bigcup_{\alpha \in A} U_\alpha$ (i.e. if $x \in E$, $x \in U_\alpha$ for some $\alpha \in A$)

2) A finite subcover of E is a finite subset $D \subseteq C$, $D = \{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ (so $\{\alpha_1, \dots, \alpha_k\} \subseteq A$) such that $E \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_k}$.

Def A set $E \subseteq S$ is compact if every open cover C has a finite subcover.

Note Every $E \subseteq S$ has a finite open cover ($\{S\}$). But E may have open cover which does in finite set such that no finite {subset} still covers E.

Example $C = \{(-1,1), (1,3)\} \cup \{(1,2), (\frac{1}{2},2), (\frac{1}{3},2), \dots, (\frac{1}{n},2)\}$ is an open cover of $[0,2]$



$D = \{(-1,1), (1,3), (\frac{1}{2}, 2)\}$ is a finite subcover.

C is also an open cover of $(0,2)$
D is also a finite subcover of $(0,2)$.

But $C' = \{(1,2), (\frac{1}{2},2), (\frac{1}{3},2), \dots, (\frac{1}{n},2)\}$ is also an open cover of $(0,2)$.

But any finite subcover $D' \subset C'$ is of the form $\{(\frac{1}{n_1},2), \dots, (\frac{1}{n_k},2)\}$. If $N > \max\{n_1, \dots, n_k\}$, $\frac{1}{N} \in (0,2)$ but $\frac{1}{N} \notin (\frac{1}{n_1},2) \cup \dots \cup (\frac{1}{n_k},2)$.