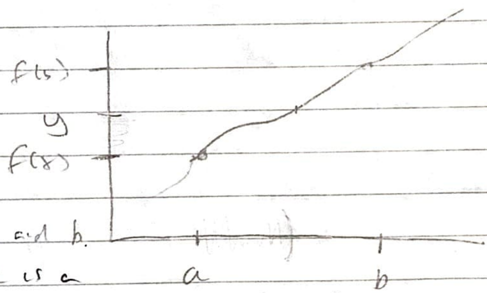
Recall an interval I has the property: if $x, y \in I$, $x < r < y$ then $r \in R$
$\Rightarrow [x,y] \subset I$

Intermediate Value Theorem Let I be an interval and $f: I \to \mathbb{R}$ be a continuous function. If $a, b \in I$, $a < b$, and y is between $f(a)$ and $f(b)$. Then there exists $x \in (a,b)$ such that $f(x) = y$.

Proof



Let $f: I \to \mathbb{R}$ be continuous, $f(a) < y < f(b)$
(The other case is the same)
Since I is an interval, $[a,b] \subset I$
Let $S = \{x \in [a,b] \mid f(x) < y\}$.
S is nonempty ($a \in S$) and bounded by a and b.
Let $x_0 = \sup S$, then $a \le x_0 \le b$ and there is a sequence (x_n) in S, $x_n \to x_0$ (choose $x_0 - 1/n < x_n \le x_0$).
Since f is continuous, $f(x_n) \to f(x_0)$. Since $x_n \in S$, $f(x_n) < y$, so $f(x_0) \le y$. Thus $x_0 \neq b$, and $x_0 < b$.

Define $y_n = x_0 + \frac{b - x_0}{n}$. Then $x_0 < y_n \le b$ and $y_n \to x_0$.

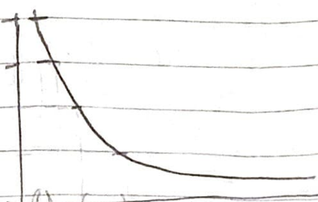
Since $y_n \notin S$, $f(y_n) \ge y$ and so $f(x_0) \ge y$. Then $f(x_0) = y$

Example $f: (0,1) \to \mathbb{R}$, $f(x) = 1/x$ is continuous (but blows up)

Let $x_0 \in (0,1)$. Let $\varepsilon > 0$. Let $\delta = \min\left(\frac{|x_0|}{2}, \frac{\varepsilon |x_0|^2}{2}\right)$
Let $x \in (0,1)$ such that $|x - x_0| < \delta$

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x||x_0|} < \frac{\delta}{(|x_0| - \delta)|x_0|} \le \frac{2\delta}{|x_0|^2} < \varepsilon$$

Since $|x| > |x_0| - \delta$ and
$|x_0| - \delta \ge |x_0|/2$.



Note δ depends on x_0, as $x_0 \to 0$
δ gets smaller

Definition $f: S \to \mathbb{R}$ is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $x, y \in S$, $|x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$

(choose δ without knowing value x or y) (only domain)

Ex $f(x) = 1/x$ is not uniformly continuous on $(0,1)$

Choose $\varepsilon = 1$. Let $\delta > 0$. Let $\delta = \min\{\delta, 1\}$.
Let $x = \sqrt{\frac{\delta}{2}}$ and $y = x - \frac{\delta}{2}$. Then $x, y \in (0,1)$, $|x-y| = \frac{\delta}{2} < \delta$,

and $\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{|x||y|} > \frac{\delta/2}{|x|^2} = 1$.

$f: [\frac{1}{2}, 1] \to \mathbb{R}$ is uniformly continuous $\delta = \min\left\{\frac{(1/2)}{2}, \frac{(1/2)^2 \varepsilon}{2}\right\}$

Theorem If $f: [a,b] \to \mathbb{R}$ is continuous, it is uniformly continuous.

Proof
Assume $f: [a,b] \to \mathbb{R}$ is continuous but not uniformly continuous.
There exists $\varepsilon > 0$ such that for all $\delta > 0$, we can find $x, y \in [a,b]$
such that $|x-y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

For each $n \in \mathbb{N}$, choose $x_n, y_n \in [a,b]$ such that $|x_n - y_n| < 1/n$, $|f(x_n) - f(y_n)| \geq \varepsilon$
Since $[a,b]$ is S. C., (x_n) has a subsequence $x_{n_k} \to x_0 \in [a,b]$
Since $|y_{n_k} - x_0| < |y_{n_k} - x_{n_k}| + |x_{n_k} - x_0| < \frac{1}{n_k} + |x_{n_k} - x_0|$, $y_{n_k} \to x_0$
Since f is continuous, $\lim_{k \to \infty} f(x_{n_k}) - f(y_{n_k}) = f(x_0) - f(x_0) = 0$
But $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$ for all k, a contradiction.

Converges, but not closed S, so continuous need not apply

$(\frac{1}{n})$ in $(0,1)$ $\to 0$, $f(x) = 1/x$, $f(\frac{1}{n}) = n$ is not Cauchy.

<u>Theorem</u> If $f: S \to \mathbb{R}$ is uniformly continuous and (x_n) is a Cauchy sequence in S, then $(f(x_n))$ is Cauchy.

<u>Proof</u> Let (x_n) in S be Cauchy. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $x, y \in S$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Choose $N \in \mathbb{R}$ such that $n, m \geq N$ implies $|x_n - x_m| < \delta$. Then $n, m \geq N$ implies $|f(x_n) - f(x_m)| < \varepsilon$. So $(f(x_n))$ is Cauchy.

<u>Thm</u> $f: (a,b) \to \mathbb{R}$ is uniformly continuous if and only if it extends to a continuous fn $f: [a,b] \to \mathbb{R}$

(i.e., add a definition of $f(a)$ and $f(b)$ to get a cont fn).

Clearly, impossible for $f(x) = 1/x$ on $(0,1)$

If f can be extended, $f: [a,b] \to \mathbb{R}$ is uniformly continuous, if $f|(a,b)$ is unif cont. (Same δ)