Lecture 10, 9/28/21
Material corresponds to Ross §17, 20.

Continuity and Limits

Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ gives a number $f(x) \in \mathbb{R}$ for every $x \in S$.

Definition (Continuity)

1. $f : S \to \mathbb{R}$ is **continuous at** $x_0 \in S$ if for every sequence (x_n) in S, $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.
2. $f : S \to \mathbb{R}$ is continuous if it is continuous at every $x \in S$.

Definition (Limit)
Given $f : S \to \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that there is a sequence in S converging to x_0, we say

$$\lim_{x \to x_0} f(x) = L$$

if for every sequence (x_n) in S, $x_n \to x_0$ implies $f(x_n) \to L$.

Theorem $f : S \to \mathbb{R}$ is continuous at $x_0 \in S$ if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

Limit and Continuity Theorems for Functions

1. Let $f, g : S \to \mathbb{R}$ be continuous $x_0 \in S$. Let $k \in \mathbb{R}$. Then $|f|, kf, f + g$, and fg are continuous at x_0. If $g(x) \neq 0$ for all $x \in S$ then f/g is continuous at x_0.
2. Let $f, g : S \to \mathbb{R}$, $\lim_{x \to x_0} f(x) = L_1$, $\lim_{x \to x_0} g(x) = L_2$. Then

$$\lim_{x \to x_0} |f(x)| = |L_1| \qquad \lim_{x \to x_0} kf(x) = kL_1$$
$$\lim_{x \to x_0} f(x) + g(x) = L_1 + L_2 \qquad \lim_{x \to x_0} f(x)g(x) = L_1 L_2.$$

   If $g(x) \neq 0$ for all $x \in S$ and $L_2 \neq 0$ then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

Definition

1. $f : S \to \mathbb{R}$, $E \subset S$, then the **image** of E is $f(E) = \{f(x) \in \mathbb{R} | \; x \in E\}$.
2. $f : S \to \mathbb{R}$, $f(S) \subseteq T$, $g : T \to \mathbb{R}$, the **composition** $g \circ f : S \to \mathbb{R}$ is defined by $g \circ f(x) = g(f(x))$.

Theorem

1. Let $f : S \to \mathbb{R}$ be continuous at $x_0 \in S$, $f(S) \subseteq T$, and let $g : T \to \mathbb{R}$ be continuous at $f(x_0) \in T$. Then $g \circ f : S \to \mathbb{R}$ is continuous at $x_0 \in S$.

2. Given $f : S \to \mathbb{R}$, $f(S) \subseteq T$, and $g : T \to \mathbb{R}$, if $\lim_{x \to x_0} f(x) = L$, $L \in T$, and g is continuous at L, then $\lim_{x \to x_0} g \circ f(x) = g(L)$.

Theorem ($\epsilon - \delta$ property)

1. $f : S \to \mathbb{R}$ is continuous at $x_0 \in S$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$.

2. Given $f : S \to \mathbb{R}$, $x_0 \in \mathbb{R}$ such that there is a sequence in S converging to x_0, $\lim_{x \to x_0} f(x) = L$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $x \in S$ and $|x - x_0| < \delta$ implies $|f(x) - L| < \epsilon$.

Examples All polynomials are continuous for all $x \in \mathbb{R}$. All rational functions are continuous where defined. $\sin : \mathbb{R} \to \mathbb{R}$ is continuous.