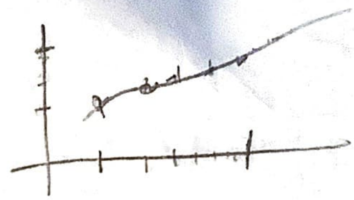

Continuity

Let $S \subseteq \mathbb{R}$. A function $f: S \to \mathbb{R}$ gives a number $f(x) \in \mathbb{R}$ for every $x \in S$. (S is part of the definition).

Def (Continuity)

1) $f: S \to \mathbb{R}$ is continuous at $x_0 \in S$ if for every sequence (x_n) in S, $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.
2) $f: S \to \mathbb{R}$ is continuous if it is continuous at every $s \in S$.

Def (limit) Suppose x_0 is the limit of a sequence in S.

Let $f: S \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. We say "$\lim_{x \to x_0} f(x) = L$" if for every sequence (x_n) in S, $x_n \to x_0$ implies $f(x_n) \to L$.

Theorem $f: S \to \mathbb{R}$ is continuous at $s \in S$ if and only if $\lim_{x \to s} f(x) = f(s)$

Limit Theorem for functions

1) Let $f, g: S \to \mathbb{R}$ be continuous at $x_0 \in S$. Let $k \in \mathbb{R}$. Then $|f|$, kf, $f+g$ and fg are continuous at x_0. If $g(x) \neq 0$ for $x \in S$, f/g is continuous at x_0.

2) Let $f, g: S \to \mathbb{R}$, $\lim_{x \to x_0} f(x) = L_1$, $\lim_{x \to x_0} g(x) = L_2$. Then

$\lim_{x \to x_0} |f(x)| = |L_1|$, $\lim_{x \to x_0} kf(x) = kL_1$, $\lim_{x \to x_0} f(x)+g(x) = L_1 + L_2$, $\lim_{x \to x_0} f(x)g(x) = L_1 L_2$

If $g(x) \neq 0$ for $x \in S$ and $L_2 \neq 0$, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$.

Proof follows from Limit Theorem for sequences.

Let (x_n) be in S such that $x_n \to x_0$. Since f, g are continuous, $f(x_n) \to f(x_0)$, $g(x_n) \to g(x_0)$. Thus $|f(x_n)| \to |f(x_0)|$, $kf(x_n) \to kf(x_0)$, $f(x_n) + g(x_n) \to f(x_0) + g(x_0)$, $f(x_n)g(x_n) \to f(x_0)g(x_0)$, and if $g(x_n) \neq 0$, $g(x_0) \neq 0$, $f(x_n)/g(x_n) \to f(x_0)/g(x_0)$. 2) is the same with $f(x_0)$ replaced by L_1, $g(x_0)$ replaced by L_2.

Def 1) $f: S \to \mathbb{R}$, $E \subseteq S$, then $f(E) = \{ f(x) \in \mathbb{R} \mid x \in E \}$ "image of E"

2) $f: S \to \mathbb{R}$, $g: T \to \mathbb{R}$, $f(S) \subset T$, the "composition" $g \circ f$ is defined by $g \circ f(x) = g(f(x))$.

Theorem If $f: S \to \mathbb{R}$ is continuous at $x_0 \in S$, $f(S) \subseteq T$, $g: T \to \mathbb{R}$ is continuous at $f(x_0) \in T$, then $g \circ f$ is continuous at x_0.

Proof
Let (x_n) be in S with $x_n \to x_0$. Then $f(x_n) \to f(x_0)$ so $g(f(x_n)) \to g(f(x_0))$.

Theorem $f: S \to \mathbb{R}$, $f(S) \subseteq T$, $g: T \to \mathbb{R}$ and $\lim_{x \to x_0} f(x) = L$, $L \in T$, and g is continuous at $L \in T$, then

$\lim_{x \to x_0} g \circ f(x) = g(L)$.

Proof Let (x_n) be a sequence in S, $x_n \to x_0$. Then $f(x_n) \to L$, so $g(f(x_n)) \to g(L)$.

Theorem ($\varepsilon - \delta$ property)
1) $f: S \to \mathbb{R}$ is continuous at $x_0 \in S$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in S$, $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

2) $f(x): S \to \mathbb{R}$, $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = L$ if and only if

for all $\varepsilon > 0$, there exist $\delta > 0$ such that $x \in S$, $|x - x_0| < \delta$ implies $|f(x) - L|$

Examples

$f:\mathbb{R}\to\mathbb{R}$ $f(x)=x$ is contn ($x_n \to x_0$, $f(x_n)=x_n \to x_0 = f(x_0)$)

$f(x)=1$ is continuous ($x_n \to x_0$, $f(x_n)=1 \to 1 = f(x_0)$)

Any polynomial is continuous. Sums of constant multiples of prods of $f(x)=x$ or $f(x)=$

Any rational func (polynom/polynom) is contn where defined.

Facts (from geometry)

$|\sin(x)| \le |x|$ $|\cos(x)| \le 1$ $\sin(x)-\sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

$\sin(\pi x)$ is contn:

Let $\varepsilon>0$, $x_0\in\mathbb{R}$. Choose $\delta=\frac{\varepsilon}{\pi}$. If $x\in\mathbb{R}$, $|x-x_0|<\delta$, then

$$|\sin(\pi x)-\sin(\pi x_0)| = 2\left|\sin\left(\pi\left(\frac{x-x_0}{2}\right)\right)\right|\left|\cos\left(\pi\left(\frac{x+x_0}{2}\right)\right)\right| \le 2\pi\frac{|x-x_0|}{2} < \pi\delta = \varepsilon.$$

Could also $\frac{2x_0-\delta}{2} \le \frac{x+x_0}{2} \le \frac{2x_0+\delta}{2} \le \frac{2x_0+1}{2}$ if $\delta<1$

($x_n \to x_0$, choose N s.t. $|x_n - x_0| < \frac{\varepsilon}{\pi}$, then $|\sin(x_n)-\sin(x_0)| < \pi\frac{\varepsilon}{\pi} = \varepsilon$.)