Homework 7

Due Tuesday, October 26 at 10am. Please upload a legible copy to bCourses.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Let $f : (a, b) \to \mathbb{R}$ be continuous. Prove that f is uniformly continuous if and only if $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$ exist.

2. Let S be a metric space with distance function d. Let S^* be a metric space with distance function d^*. Prove that a function $f : S \to S^*$ is continuous if and only if for ever closed set $E \subset S^*$, $f^{-1}(E)$ is also closed.

3. Prove that $\mathbb{R}$ is not compact. Note: use the definition of compactness; do not use sequential compactness.

4. Let S be a metric space with distance function d. Let E be a compact subset of S. Let F be a closed subset of S such that $F \subseteq E$. Prove that F is compact. Note: use the definition of compactness; do not use sequential compactness.

5. Let $I \subset \mathbb{R}$ be a bounded set with the interval propery (if $a, b \in I$ and $a < c < b$ then $c \in I$.)

   a) If I is closed, prove that $I = [x, y]$ for some $x, y \in \mathbb{R}$. What are x, y?

   b) If I is open, prove that $I = (x, y)$ for some $x, y \in \mathbb{R}$. What are x, y?

6. Let S be a metric space with distance function d. Let E and F be connected subsets of S such that $E \cap F$ is nonempty. Prove that $E \cup F$ is connected.

7. Prove that the "unit circle" $E = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ is connected. Hint: recall that sin and cos are continuous and parameterize.