Homework 6

Due Tuesday, October 12 at 10am. Please upload a legible copy to bCourses.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Let $a > 1$. Prove that $f : \mathbb{R} \to \mathbb{R}, f(x) = a^x$ is continuous. You may use the following facts without proof:
   - $a^0 = 1$
   - $a^{x+y} = a^x a^y$
   - If $x < y$ then $a^x < a^y$
   - Ross Theorem 9.7, (d)

2. Ross 17.12 Hint: use the density of the rationals $\mathbb{Q}$

3. Ross 17.13 Hint: also use the density of the irrationals $\mathbb{R}\backslash\mathbb{Q}$

4. Prove that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ in not uniformly continuous.

5. Let $a_0, a_1, a_2, a_3 \in \mathbb{R}$ with $a_3 \neq 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Prove that there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let (x_n) be a Cauchy sequence. Prove that $(f(x_n))$ is a Cauchy sequence.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.