Homework 3

Due Tuesday, September 21 at 10am. Please upload a legible copy to bCourses.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Ross 8.2 a), c) and e). This time, use limit theorems (Ross Theorems 9.2-9.6, as well as the result you proved in Ross Problem 8.5.)

2. a) Let $a < 1$. Prove that $1 + a + ... + a^n = \frac{1-a^{n+1}}{1-a}$
   b) Let (s_n) be a sequence in $\mathbb{R}$ such that $|s_{n+1} - s_n| < 1/2^n$ for all n. Prove that s_n is Cauchy.

3. Ross 10.7: Let S be a bounded nonempty subset of $\mathbb{R}$ and suppose $\sup S \notin S$. Prove that there is a nondecreasing sequence (s_n) of points in S such that $\lim s_n = \sup S$.

4. Assume that $s_n \to s$. Prove that $\liminf s_n \geq s$. (This was a step in the proof that a sequence converges if and only if $\limsup = \liminf$).

5. Let s_n be defined inductively by $s_1 = 2$ and $s_{n+1} = \frac{s_n}{2} + \frac{1}{s_n}$.
   a) Show that $s_n^2 - 2 > 0$ for all $n \in \mathbb{N}$.
   b) Prove that s_n is monotone. Prove that s_n converges.

6. Let s_n be the sequence defined in the previous problem.
   a) Find $\lim s_n$ and justify your answer. (Hint: We know (s_n) converges. For large n, s_n and s_{n+1} are very close to $\lim s_n$. What equation must $\lim s_n$ satisfy?)
   b) Conclude that there are Cauchy sequences in $\mathbb{Q}$ which do not converge to any $s \in \mathbb{Q}$.

7. For each sequence (s_n) below, find the following and justify your answer:
   - $\liminf s_n$ and a monotone subsequence (t_k) of (s_n) such that $t_k \to \liminf s_n$.
   - $\limsup s_n$ and a monotone subsequence (r_l) of (s_n) such that $r_l \to \limsup s_n$

   a) $s_n = (-1)^n$
   b) $s_n = (-1/2)^n$
   c) $s_n = (-1)^n + 1/n$