Homework 12

Due Thursday, December 9 at 10am. Please upload a legible copy to bCourses.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. For each $n \in \mathbb{N}$, define $f_n : (-1,1) \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$.
   a) Prove that (f_n) converges uniformly to $f(x) = |x|$.
   b) Prove that f_n is differentiable and find f_n'.
   c) Find the function $g : (-1,1) \to \mathbb{R}$ such that (f_n') converges pointwise to g. Prove that (f_n') does not converge uniformly to g.

2. Prove that $\sum_{n=1}^{\infty} nx^n$ converges to $\frac{x}{(1-x)^2}$ for $x \in (-1,1)$. Hint: (Use the fact that $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$).

3. Use the fact that for each $x \in \mathbb{R}$, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ to prove that $(e^x)' = e^x$.

4. Define $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Find a power series which converges at each $x \in \mathbb{R}$ to $\int_0^x f$.

5. Prove that there does not exists a power series which converges pointwise to $f : (-1,1) \to \mathbb{R}$, $f(x) = |x|$

6. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-\frac{1}{x^2}}$ when $x \neq 0$ and $f(0) = 0$. Prove that when $x \neq 0$

$$f^{(n)}(x) = e^{-\frac{1}{x^2}} \sum_{k=1}^{3n} \frac{a_{n,k}}{x^k}$$

   for some constants $a_{n,k} \in \mathbb{R}$. (Hint: Use induction on n, with base case $n = 1$. You do not need to find formulas for the $a_{n,k}$)

7. Let f be the same function as in the previous problem.
   a) Prove that, for any $k \in \mathbb{N}$,

$$\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = 0.$$

   (Hint: Substitute ($y = \frac{1}{x}$) and apply L'Hospital's rule several times. You do not need to justify all the hypothesis of L'hospital's rule each time)
   b) Prove that for all $n \in \mathbb{N}$

$$\lim_{x \to 0} \frac{f^{(n)}(x)}{x} = 0.$$

   c) Prove that $f^{(n)}(0) = 0$ for all n (Hint: Use induction, with base case $n = 0$. Use part c and the definition of the derivative.)
   d) What is the Taylor series for f centered at 0? At what $x \in \mathbb{R}$ does it converge to $f(x)$?