Homework 11

Due Thursday, December 2 at 10am. Please upload a legible copy to bCourses.

You may work together, but the solutions must be written up in your own words. Show all work and justify all answers.

1. Let $f_n, f : S \to \mathbb{R}$ be functions, for all $n \in \mathbb{N}$. Prove that $f_n \to f$ uniformly if and only if
$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| \mid x \in S\} = 0.$$

2. For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \left(x - \frac{1}{n}\right)^2$. Find $f : [0,1] \to \mathbb{R}$ such that $f_n \to f$ uniformly, and prove your assertion.

3. For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = nx^n(1-x)$. Let $f : [0,1] \to \mathbb{R}$ be $f(x) = 0$. Prove that $f_n \to f$ pointwise, but not uniformly. Recall $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

4. Ross 25.7

5. Define $g_k : (0,1) \to \mathbb{R}$ by $g_k(x) = x^k$. Prove that $\sum_{k=0}^{\infty} g_k$ converges pointwise to $f(x) = \frac{1}{1-x}$ (that is, the sequence of partial sums converges pointwise). Prove that $\sum_{k=0}^{\infty} g_k$ does not converge uniformly.

6. Ross 25.3

7. Ross 23.1 a), c), e), g).