Practice Problems

These problems mostly cover the second half of the semester. They do not cover every topic, and are meant as extra practice, not a comprehensive review.

1. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that $\int_a^b f = \int_a^b g$. Prove that $f(x) = g(x)$ for some $x \in [a, b]$.

2. Find the Taylor series for $f : (-1, 1) \to \mathbb{R}$, $f(x) = \sqrt{1 - x}$. Use Taylor's theorem to prove that the Taylor series converges to f.

3. Define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{x}{1+nx^2}$. Find $f : \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ uniformly. Prove that $\lim_{n\to\infty} f_n'(x) = f'(x)$ for all $x \in \mathbb{R}$ except $x = 0$.

4. For each $k \in \mathbb{N}$ define

$$g_k : [-\pi, \pi] \to \mathbb{R} \text{ by } g_k(x) = (\sin(x))^2(\cos(x))^{2k}.$$

a) Find $f : [-\pi, \pi] \to \mathbb{R}$ such that $\sum g_k \to f$ pointwise.

b) Does $\sum g_k \to f$ uniformly?

c) If we change the domain to $[\pi/4, 3\pi/4]$, does $\sum g_k \to f$ uniformly?

5. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{\sin(x)}$ and $g(x) = \int_0^{x^2} f$. Find $g'(x)$.

6. Let $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ such that $\sum a_n 2^n$ converges. Prove that the sequence of functions $\sum a_n x^n$ converges uniformly on $[-1, 1]$.

1) Define $F : [a,b] \to \mathbb{R}$

$$F(x) = \int_a^x f - g .$$

Then $F(a) = F(b) = 0$.

Since $f-g$ is continuous, by the Fund. Thm. part II

- F is cont. on $[a,b]$
- F is differentiable on (a,b)
- $F'(x) = f(x) - g(x)$ for $x \in (a,b)$.

By the MVT, there exists $x_0 \in (a,b)$ such that

$$F'(x_0) = f(x_0) - g(x_0) = \frac{F(b) - F(a)}{b-a} = 0.$$

<u>Alternate Solution</u>

If $f(x) \geq g(x)$ for all $x \in [0,1]$ or $g(x) \geq f(x)$ for all $x \in [0,1]$ then $\int_a^b f-g = \int_a^b g-f = 0$ implies $f(x) = g(x)$ for all $x \in [0,1]$ since $f-g$ is continuous.

If neither case holds, then there exist x_1 such that $f(x_1) > g(x_1)$ and x_2 such that $g(x_2) > f(x_2)$.
Applying the intermediate value theorem to $f - g$ there exists x_0 between x_1, x_2 where $f(x_0) = g(x_0)$.

2) $f'(x) = -\frac{1}{2}\frac{1}{(1-x)^{1/2}}$ $f''(x) = -\left(\frac{1}{2}\right)^2 \frac{1}{(1-x)^{3/2}}$

$$f^{(3)}(x) = -\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2 \frac{1}{(1-x)^{5/2}}$$

$$f^{(4)}(x) = -\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2 \frac{1}{(1-x)^{7/2}}$$

$n \geq 2$

$f^{(n)}(x) =$

$$-\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\cdots\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\frac{1}{(1-x)^{\frac{2n-1}{2}}}$$

Taylor Series

$$-\frac{1}{2}x - \sum_{n=2}^{\infty}\frac{(2n-3)(2n-5)\dots(1)}{2^n n!}x^n$$

Let $x \in (-1, 0)$.
By Taylor's Thm, there exists $y \in (x, 0)$ such that $R_n(x) = \frac{f^{(n)}(y)}{n!}x^n$.

Note that

$$(2n-3)(2n-5)\cdots(3)(1) = \frac{(2n-3)!}{(2n-4)(2n-6)\cdots(4)(2)}$$

$$= \frac{(2n-3)!}{2^{n-2}(n-2)(n-3)\cdots(2)(1)} = \frac{(2n-3)!}{2^{n-2}(n-2)!}$$

So, since $-1<x<y<0$

$$|R_n(x)| = \left| \frac{(2n-3)!}{2^{2n-2}(n-2)!\,n!} \frac{x^n}{(1-y)^{\frac{2n-1}{2}}} \right| \leq \frac{(2n-3)!}{2^{2n-2}(n-2)!\,n!} |x|^n = a_n$$

We show that $a_n \to 0$. Indeed,

$$\frac{a_{n+1}}{a_n} = \frac{(2n-1)!\,|x|^{n+1}}{2^{2n}(n-1)!(n+1)!} \frac{2^{2n-2}(n-2)!\,n!}{(2n-3)!\,|x|^n} = \frac{(2n-1)(2n-2)\,|x|}{4(n-1)(n+1)} \to |x|<1$$

So $\sum a_n$ converges, which in turn implies $a_n \to 0$.

So $R_n(x) \to 0$ for all $x \in (-1,0)$.

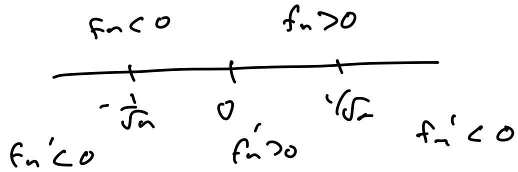
3)

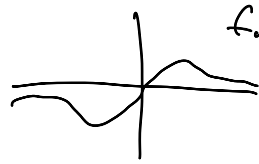
Let $f(x) = 0$ for all $x \in \mathbb{R}$.

$$f_n'(x) = \frac{1+nx^2 - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

$f_n'(x) = 0$ exactly when $x = \pm\frac{1}{\sqrt{n}}$.

Analysis of the sign of f_n, f_n' shows f_n achieves its min at $-\frac{1}{\sqrt{2}}$, max at $\frac{1}{\sqrt{2}}$





So $\sup\{|f_n(x)| \mid x \in \mathbb{R}\} = |f_n(\pm\frac{1}{\sqrt{n}})| = \frac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$. Thus $f_n \to f$ uniformly.

$f' = 0$. If $f_n'(0) = 1 \to 1 \neq f'(0)$ as $n \to \infty$ but

If $x \neq 0$, $f_n'(x) = \frac{1/n^2 - x^2/n}{(\frac{1}{n} + x^2)^2} \to 0 = f'(0)$.

4) a) If $x \in \{\pi, -\pi, 0\}$, $\sin(x) = 0$,

so $\sum g_k(0) = 0$

otherwise, $|\cos(x)| < 1$, so the

geometric series $\sum_{k=0}^{\infty} (\cos^2(x))^k = \frac{1}{1-\cos^2(x)} = \frac{1}{\sin^2(x)}$

and $\sum_{k=0}^{\infty} g_k(x) = \sin^2(x) \sum_{k=0}^{\infty} (\cos^2(x))^k = 1$.

So $f(x) = \begin{cases} 0 & x \in \{\pi, -\pi, 0\} \\ 1 & x \notin \{\pi, -\pi, 0\} \end{cases}$.

b) Since each g_k is continuous, if the convergence were uniform, f would be continuous. So the convergence is not uniform.

c) For $x \in [\pi/4, \frac{3\pi}{4}]$, $|\cos(x)| \leq \frac{1}{\sqrt{2}}$

so $|g_k(x)| \leq \frac{1}{2^k}$. Since $\sum \frac{1}{2^k}$ converges, the M test implies $\sum g_k$ converges uniformly.

5) Define $F : \mathbb{R} \to \mathbb{R}$ by $F(y) = \int_0^y f$. Since f is continuous,

F is differentiable and $F' = f$ by the fundamental thm. Let $u(x) = x^2$.

Then $g = F \circ u$ so

$$g'(x) = F'(u(x))\, u'(x) = e^{\sin(x^2)}(2x).$$

6) Let R be the radius of convergence of $\sum a_n x^n$. If $R < 2$, then $\sum a_n 2^n$ does not converge. So $R \geq 2$.

Since $1 < 2$, $\sum a_n x^n$ converges uniformly on $[-1, 1]$.