Aspects of Gauge Theories

Problem 1. S-duality of abelian gauge theory

Read section two of Witten’s paper “On S-duality in Abelian Gauge Theory”.

Consider a Euclidian $U(1)$ gauge theory in $d = 4$ on $T^4$ with flat metric and coupling constant

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

with the action given by

$$S = \frac{1}{g^2} \int F \wedge *F + \frac{i\theta}{8\pi^2} \int F \wedge F$$

i. Show that the path integral splits to various topological sectors related to the first Chern class of the bundle.

ii. Show that for each class the path integral can be computed from a classical contribution and an infinite dimensional Gaussian integral $Z_{qu}$.

iii. Show that for a particular value of $Z_{qu}$ the path integral is invariant under $SL(2, \mathbb{Z})$ symmetry

$$\tau \to (a\tau + b)/(c\tau + d).$$

Consult Witten’s paper to justify this choice of $Z_{qu}$.

iv. Recall that

$$F = \frac{1}{2} F_{ij} dx^i \wedge dx^j$$

is related to the electric field $\vec{E}$ and magnetic field $\vec{B}$ by

$$F = E_1 dx^1 \wedge dx^4 + E_2 dx^2 \wedge dx^4 + E_3 dx^3 \wedge dx^4 + B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2$$

(the $x^4$ direction corresponding to Euclidian time.) Show that $\tau \to -1/\tau$ symmetry exchanges electric and magnetic degrees of freedom.

For $\theta = 0$ this exchanges $g_{YM}$ and $1/g_{YM}$ – it is a strong-weak coupling duality. In a bosonic Yang-Mills theory based on a non-abelian lie group $G$ this symmetry is broken by quantum effects. However, the Yang-Mills theory with $N = 4$ supersymmetry is believed to have S-duality which relates the theory with gauge group $G$ with one based on the “Langlands dual” group $\hat{G}$. 