Problem Set 3  

Due: Feb. 23, 2005

QFT in $1 + 1$ dimensions: Perturbative String Theory

**Problem 1. 2d sigma model with flat target spaces**

In this problem, we will consider 2d sigma models on $T^2$, with flat target space $X$. The $T^2$ is defined as a quotient of the complex plane by

$$z \sim z + 2\pi \sim z + 2\pi \tau.$$

Here we will develop the path integral formulation to compare with the operator formulation discussed in class.

i. Consider the target space to be $X = R$. What is the sigma model action? Start by finding the metric on $T^2$.

ii. Compute the path integral as a function of $\tau$. Does this agree with the results found in class in the operator formalism?

iii. Consider now $X = S^1$ of radius $R$. Compute the partition function. Show that this has a T-duality symmetry. (You may need Poisson resummation for comparisons, as in problem number 1 in problem set 1. Be careful regarding the zero modes, and how the measure changes with any changes of variables.) Note that all the $R$-dependence is in the classical part. Why is that?

iv. Generalize iii. to $X = T^n$ with metric

$$G_{I,J}dX^I dX^J$$

where $X^I \sim X^I + 1$, and $G_{I,J}$ is a constant, positive and non-degenerate symmetric matrix. Be careful regarding the measure.

v. *Turning on B-field.* Consider now turning on a flat anti-symmetric 2-form

$$B = \frac{1}{2} B_{ij} dX^i \wedge dX^j.$$
Compute the partition function in operator formalism (Be careful to correctly identify the momentum conjugate to $X^I$), and in the path integral formalism.

Do they agree?

**Problem 2. T-duality another way**

Consider a path integral on a fixed genus $g$ Riemann surface $\Sigma$, with the target space $X = T^n$. Denote the partition function by $Z(\Sigma, G)$ where $G$ denotes the flat metric on $X$.

i. By direct evaluation of the path integral, but without attempting to compute the determinants involved, show that the T-duality is a symmetry of the path integral, up to an overall factor involving the volume of $X$. This will allow you to determine the relative normalization of the path integrals relating the original and the dual theory, which I avoided in the class.

ii. In string perturbation theory (where one in addition integrates over the moduli space of Riemann surfaces), the genus $g$ surface is weighted by a factor $\lambda^{2g-2}$ where $\lambda$ denotes the string coupling constant. Show that the overall factor found in part i.) can be incorporated in the change of string coupling.

**Problem 3. Gauge theory in 0 + 1 dimension**

Consider a $U(1)$ gauge field $A$, i.e. a connection on a line bundle, on a circle of circumference $\beta$. Consider the action

$$S = i \alpha \oint_{S^1} A$$

i. Show that $exp(-S)$ is gauge invariant, i.e. that it does not change under

$$A \rightarrow A + g^{-1}dg$$

where $g : S^1 \rightarrow U(1)$, only if $\alpha$ is an integer $k$

ii. Show that the above action does not depend on $\beta$, in other words, it does not depend on the metric on $S^1$. This is a simple example of a topological field theory,

iii. Compute the path integral

$$Z = \int DA \exp(-S).$$