QFT in $1 + 1$ dimensions: Perturbative String Theory

Problem 1. Symmetries and Conservation Laws

In class, we discussed the Noether method which, to a symmetry of the theory, associates a divergence free current, and a “charge” which generates this symmetry via Poisson brackets in the classical theory. As in the class, consider a 2d sigma model corresponding to maps $R \times S^1 \to R$, with the action

$$S(X) = -\frac{1}{4\pi} \int d^2\zeta \sqrt{-g} g^{\alpha\beta} \partial_\alpha X \partial_\beta X$$

where $g_{\alpha\beta}$ is the flat metric on the cylinder with Minkowski signature, $g = \text{diag}(-1, +1)$.

i. Show that time translations and rotations around the $S^1$ are symmetries of the theory. That is, if $\zeta^0 = t, \zeta^1 = \theta$ are coordinates on the cylinder in parameterizing the $R$ and the $S^1$ directions, show that the action is invariant under

$$\zeta^\alpha \to \zeta^\alpha + \epsilon^\alpha$$

for a constant $\epsilon$, independent on $\zeta^\alpha$.

ii. Use the Noether procedure derived in class to find the corresponding conserved currents. Assuming that $\epsilon$ is small, but not constant, show that the action varies as

$$\delta_\epsilon S = -\frac{1}{4\pi} \int d^2\zeta \partial_\beta \epsilon^\alpha T^\beta_\alpha [X(\zeta)],$$

and compute $T^\beta_\alpha$.

iii. Provide a general argument that $T^\beta_\alpha$ is divergence free when evaluated on classical equations of motion, in the sense that

$$\partial_\beta T^\beta_\alpha [X_{cl}(\zeta)] = 0$$

$^1$ The repeated indices are summed over. By $A_\alpha B^\alpha$, I mean $\sum_{\alpha=t,\theta} A_\alpha B^\alpha$. 

The object $T^\alpha_\beta$ is called the “energy-momentum tensor”.

iv. Use the result of ii.), iii.) to show that the quantity

$$Q_\alpha = \int d\theta \, T^\alpha_\theta$$

is conserved on solutions to classical equations of motion.

$$\frac{d}{dt} Q_\alpha [X_{cl}(\zeta)] = 0.$$ 

v. We now want to show that $Q_t$ agrees with the hamiltonian $H$ of the theory,

$$Q_t = H$$

and therefore corresponds to generator of time-translations, as claimed. Starting with the Lagrangian density $\mathcal{L}$ of the theory, compute the momentum $\Pi(\zeta) = \frac{\delta \mathcal{L}}{\delta \partial_t X}$ conjugate to $X(\zeta)$. Use this to compute the hamiltonian density $\mathcal{H}$, and the corresponding Hamiltonian

$$H = \int d\theta \, \mathcal{H}.$$ 

Does this agree with $Q_t$?

vi. In class, we denoted $Q_\theta = P$. Here we want to show that indeed it corresponds to a generator of translations along the $S^1$. We’ll do so as follows. Recall that in quantum theory $P$ is associated to a Hermitian operator $\hat{P}$. Consider the corresponding unitary operator

$$\hat{U}(\alpha) = \exp(-i\alpha \hat{P}).$$

Show that

$$\hat{U}^{-1}(\alpha) \hat{X}(t, \theta) \hat{U}(\alpha) = \hat{X}(t, \theta + \alpha).$$

Problem 2. Virasoro Algebra

Consider the 2d QFT for a string moving on $R^d$. Define the operators

$$L_n = \frac{1}{2} \sum_{i=1}^d \sum_{k=-\infty}^{\infty} : \alpha_i^{n-k} \alpha_i^k :$$

where the notation is as in the lecture, with:

$$[\alpha_n^i, \alpha_m^j] = n \delta_{n+m,0} \delta^{i,j}$$

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and : ... : denotes an ordering of α’s that puts the raising operators on the left side, and lowering operators on the right side (this is called “normal ordering”).

i. Show that

\[ [L_n, L_m] = (n - m)L_{n+m} + \text{const}.(n^3 - n)\delta_{n+m,0} \]

and determine the constant in the above expression. Note that \( L_0 = H_L \) and we can similarly define another copy of the Virasoro algebra using \( \tilde{\alpha} \) operators, with \( \tilde{L}_0 = H_R \).

ii. What is the geometric meaning of the above algebra on the field \( X(z, \bar{z}) \), where \( z = \exp(t + i\theta) \)? Hint: Consider the effect on \( X \) of holomorphic change of variables \( z \to z' = f(z) \).

iii. Show that the \( L_m, \tilde{L}_m \) are modes of energy momentum tensor \( T_{\alpha\beta} \). \( T \) is called a tensor since the object invariant under coordinate transformations is

\[ T_{\alpha\beta}d\zeta^\alpha d\zeta^\beta \]

so \( T_{\alpha\beta} \) is not a function (but a section of \( (T^*)^2 \)). Show that \( T_{z\bar{z}} \) vanishes, and relate \( L_m \)'s and \( \tilde{L}_m \)'s to modes of \( T_{zz} \) and \( T_{\bar{z}\bar{z}} \).

**Problem 3. Topological Quantum Field Theory**

Consider a two-dimensional quantum field theory on an oriented Riemannian manifold \( \Sigma \) with boundaries. We’ll denote by \( \Sigma_{g,n} \) a surface of genus \( g \) with \( n \) boundaries. As discussed in class, the path integral on \( \Sigma_{g,n} \) defines a state \( |\psi(\Sigma_{g,n})\rangle \) in the n-fold tensor product of Hilbert space:

\[ |\psi(\Sigma_{g,n})\rangle \in \mathcal{H}^{\otimes n-k} \otimes \mathcal{H}^* \]

where we have \( n - k \) boundaries of one orientation and \( k \) of opposite. In a topological quantum field theory, the state \( |\psi(\Sigma)\rangle \) does not depend on the choice of metric on \( \Sigma \). Assume, as is sometimes the case, that the Hilbert space of the theory is finite dimensional.

i. Show that path integral on a disk with an outgoing boundary, picks out a distinguished state \( e_0 = |0\rangle \) in \( \mathcal{H} \). Correspondingly a path integral on a disk with an incoming boundary picks out a dual state \( e_0^* = \langle 0| \in \mathcal{H}^* \).

ii. Show that, for \( \Sigma_{0,2} \) with boundaries of opposite orientation, the state \( |\psi(\Sigma_{0,2})\rangle \) can be viewed as an operator mapping \( \mathcal{H} \to \mathcal{H} \). Show moreover that this operator is the identity.
iii. **State-Operator Correspondence.** Choose a basis $e_i$ of $\mathcal{H}$. For $\Sigma_{0,3}$ with 2 incoming and one outgoing boundary, show that to each $e_i \in \mathcal{H}$ we can associate an operator

$$\hat{e}_i : \mathcal{H} \to \mathcal{H}$$

which in coordinate basis can be characterized by matrices

$$(\hat{e}_i)^k_j = e^k_{ij}.$$ 

Show that $e^k_{0j} = \delta^k_j$. Moreover, show that $e^0_{ij} = \eta_{ij}$ determines an inner product on $\mathcal{H}$:

$$\eta : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$$

and thus also provides an isomorphism between $\mathcal{H}$ and $\mathcal{H}^*$.

iv. Show that $e^k_{ij} = e^k_{ji}$.

v. Compute $\Sigma_{0,4}$ partition function in terms of $e^k_{ij}$. What consistency conditions follow from sewing properties?

vi. Define an operator by considering the path integral for $(g, n) = (1, 2)$ This is called the Handle operator $H$. Compute the partition function of $\Sigma_{g,0}$ in terms of $H$.

vii. Compute the path integral for arbitrary $(g, n)$ in terms of $e^k_{ij}$. Are there any other consistency conditions on $e^k_{ij}$ for the path-integral to be well defined?