QFT in 0 + 0 and 0 + 1 dimensions

Problem 1. QFT in 0 + 0 dimensions

The path integral for the most trivial QFT is just an integral over some variable, with an action defined in terms of those variables. Consider a "$\phi^4$ theory": the case of having only one "field" – a variable $\phi$ with the action

$$S(\phi) = \alpha \phi^2 + \epsilon \phi^4.$$ 

and consider the path integral which in this case is just the ordinary integral

$$Z(\alpha, \epsilon) = \int_{-\infty}^{\infty} d\phi \exp(-S(\phi)).$$

Assume $\epsilon$ is small. The aim of this problem is to define a perturbative expansion in $\epsilon$, as means of computing $Z(\alpha, \epsilon)$.

i. What is $Z(\alpha, 0)$?

ii. Consider corrections to i). Expand

$$exp(-\epsilon \phi^4) = \sum_n (-\epsilon \phi^4)^n / n!.$$

The $n$-th term in the expansion is called the $n$-th order perturbative correction to $Z(\alpha, 0)$. Compute the perturbative corrections up to second order in $\epsilon$ by evaluating the corresponding integrals.

iii. We wish to associate a graph to each order in perturbative correction: To each term $\epsilon \phi^4$ associate a quartic vertex of a graph (a point with 4 lines ending on it). Can you interpret the results in ii.) in terms of making graphs? What factors do you associate to each graph?

iv. Formulate the perturbative corrections to all orders in terms of summing over graphs, each with a particular weight.
**Problem 2. Path Integral and Operator Formalism of Quantum Mechanics**

In the operator formalism of quantum mechanics, the amplitude for particle to propagate from \(x = x_0\) at time \(t\) to \(x = x_1\) at \(t = t + \Delta t\) is given by

\[
\langle x_1 | \exp(-i\hat{H}\Delta t/\hbar) | x_0 \rangle \tag{1}
\]

where \(\hat{H}\) is the Hamiltonian operator, \(|x\rangle\) is an eigenstate of operator \(\hat{x}\), e.g. \(\hat{x}|x_0\rangle = x_0|x_0\rangle\). The aim of this problem is to show that this is equal to the following path integral

\[
\int_{x(t) = x_0}^{x(t+\Delta t) = x_1} \mathcal{D}x \exp\left(-i \int_{t' = t}^{t' = t+\Delta t} L \, dt' / \hbar\right) \tag{2}
\]

taken over all paths with the boundary conditions \(x(t) = x_0\), \(x(t + \Delta t) = x_1\), where \(L\) is the Lagrangian.

\(i\). Consider a particle of unit mass moving in potential \(V(x)\) on a real line, parameterized by \(x\). What are the Lagrangian \(L\) and the Hamiltonian \(H\) for this system?

\(ii\). Assuming the time interval \(\Delta t = \epsilon\) is infinitesimal, compute the amplitude in (1) to the first order in \(\epsilon\). (Hint: Evaluating this purely in position space may be awkward. Insert the resolution of identity \(id = \int dp \langle p | \rangle \langle p |\), and write the result in terms of an integral over \(p\).)

\(iii\). Use the result of \(i\). to evaluate the propagator for a finite value of \(\Delta t\). Divide the interval into \(N\) intervals of size \(\epsilon\), \(\Delta t = N\epsilon\), where \(N\) is large, so that \(\epsilon\) is small. Propagate the particle in steps: from \(x(t) = x_0\), to \(x(t + \epsilon) = x_{0,1}\), from there to \(x(t + 2\epsilon) = x_{0,2}\), and so on to \(x(t + N\epsilon) = x_1\). This defines a discretized path passing through \(x_0, x_{0,1}, \ldots, x_{0,N} = x_1\).

![Fig.1 A discretization of the time interval \(\Delta t\) into \(N\) slices, and a the discretization of a path.](image-url)
To get an amplitude corresponding to (1) integrate over $x_{0,1} \ldots x_{0,N-1}$. This is a path integral over discretized paths (in the operator language, what we have done is inserted $N - 1$ resolutions of identity $\int dx_{0,i}|x_{0,i}\rangle \langle x_{0,i}|$.) Take $N \to \infty$. What is the path integral that you get? (See iv.)

iv. The path integral you should get involves path-integration over two variables $x$ and $p$. Note that the path integral over $p$ has no boundary conditions specified. This is a reflection of the fact that $p$ and $x$ are canonically conjugate variables, and one cannot fix both $x$ an $p$ on the boundaries. Compute the integral over $p$ explicitly. Does the resulting path integral over $x$ agree with (2)?

**Problem 3. 0 + 1 dimensional sigma model**

Consider a free particle of unit mass, moving on a circle of radius $R$.

i. What is a suitable Hilbert space for this problem?

ii. What is the Hamiltonian $H$ for this system? Find the spectrum of the Hamiltonian, and the corresponding eigenvectors.

iii. Compute the partition function

$$Z(\beta) = Tr \exp(-\beta H)$$

iv. Formulate a path-integral computation which is equivalent to computing $Z(\beta)$ as certain integrals over the space of maps from a circle of circumference $\beta$ to a circle of radius $R$.

v. Evaluate the path integral by a saddle point expansion near the stationary points of the action (Hint: The space of saddle points is indexed by an integer).

vi. Using Poisson resummation, show that the answer in v.) is equivalent to that found in iii.).

vii. Sigma model on $X$. Now generalize to a 0 + 1 dimensional QFT with target space a Riemannian manifold $X$, i.e. consider maps $S^1 \to X$. What is the Hamiltonian in this case? What it the definition of the path integral?