TOPOLOGICAL STRINGS,

MONODROMY,

AND

(ALMOST) MODULAR FORMS

hep-th/0607100

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Symmetry is key to any attempt to understand a theory.

**Symmetry**

in

**The Topological String**

**A-Model**

Counting of holomorphic maps to $X$

**B-Model**

Variations of complex structures of $Y$

$X, Y =$ Calabi-Yau 3-folds
Many properties of the theory, in particular the symmetries, become more transparent when moduli of $x_1, y$ are allowed to vary:

The B-model is ideally suited for this.

Recall some basic facts about this theory.
Given a symplectic basis of $H_3(Y)$

$$A^2 \wedge B_0 = \delta^2$$

And a choice of a 3-form $\omega$,

the periods

$$x^2 = \delta^2 \omega, \quad p_i = \delta^2 \omega_{B_i}$$

are not independent:

$$p_i = \frac{2}{2x_i} F_0$$

Genus zero topological string amplitude

$H^3(Y) = \text{phase space}$

with symplectic form $dp_i \wedge dx^i$
IN THE QUANTUM THEORY, $\mathbf{p}_j, X^j$ BECOME OPERATORS

$$[p_j, X^j] = g_s^2 \delta^j_z$$

AND, TOPOLOGICAL STRING PARTITION FUNCTION, A WAVE FUNCTION

$$\mathcal{Z}(x) = \langle x | \mathcal{Z} \rangle$$

$$\mathcal{Z}(x) = \exp \left( \sum_{g_s \to 0} g_s^{2g-2} \mathcal{F}_g(x) \right)$$

$$= \exp \left( g_s^{-2} \mathcal{F}_1 + g_s^{-1} \mathcal{F}_2 + g_s^{0} \mathcal{F}_3 + \ldots \right)$$

$$\partial \mathcal{Z}(x) = g_s^2 \frac{\partial}{\partial x} \mathcal{Z} \sim \frac{2}{\partial x} \mathcal{F}_0 \text{ as } g_s \to 0$$
Different choices of basis for $H_3(Y)$ are related by symplectic transformations

\[
\begin{pmatrix}
    p \\
    x
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    \tilde{p} \\
    \tilde{x}
\end{pmatrix}
= \begin{pmatrix}
    A & B \\
    C & D
\end{pmatrix}
\begin{pmatrix}
    p \\
    x
\end{pmatrix}
\]

\[
\begin{pmatrix}
    A & B \\
    C & D
\end{pmatrix}
\in Sp(2n, \mathbb{Z}).
\]

This acts on the wave function

\[
\tilde{Z}(\tilde{x}) = \int dx \, e^{-S(x, \tilde{x})} / q_i^2 \, Z(x)
\]

\[
S(x, \tilde{x}) = -\frac{1}{2} x^T (C^T D) x + x^T C^{-1} \tilde{x} - \frac{1}{2} \tilde{x}^T (AC^T) \tilde{x}
\]

for $C$ invertible.
For a subgroup

\[ \Gamma \subset \text{Sp}(2n, \mathbb{Z}) \]

This is a symmetry: the change of basis can be undone by picking a different \( W \).
EXAMPLE:

THE MIRROR OF QUINTIC CY 3-FOLD:

\[ \sum_{\varphi=1}^{5} x^\varphi - 5 \psi \prod_{\varphi=1}^{5} x^\varphi = 0, \quad \text{IN } \mathbb{P}^4 \]

MODULI SPACE:

\[ \psi = 0 \quad \psi = \infty \]
\[ \varphi = 1 \quad Z = \frac{1}{\psi^5} \]

THE SYMMETRY GROUP \( \Gamma \) IS GENERATED BY MONODROMIES

\[ M_0, M_1, M_\infty \]
In terms of a natural basis of periods

\[ \Pi = \begin{pmatrix} p_1 \\ p_0 \\ x^1 \\ x^0 \end{pmatrix} \sim \begin{pmatrix} 4\text{-cycle} \\ 6\text{-cycle} \\ 2\text{-cycle} \\ 0\text{-cycle} \end{pmatrix} \]

\[ \Psi = 0 \]

\[ x = \frac{1}{\Psi^5} \]

\[ \Pi(\alpha \Psi) = M_0 \Pi(\Psi), \quad \alpha^5 = 1 \]

\[ M_0 = \begin{pmatrix} -19 & -3 & 5 & -3 \\ 31 & -4 & -8 & -5 \\ -80 & -11 & 21 & -11 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_0^5 = \text{id} \]
WHAT DOES THIS MEAN FOR THE PARTICLE FUNCTION \( Z(x) \)?

\( Z \) IS, AT LEAST LOCALLY, A FUNCTION ON THE MODULI SPACE \( Z(x^i(\psi)) = Z(\psi) \)

\( \psi \rightarrow M_\nu \psi \)

\( Z(\psi) \rightarrow Z(M_\nu \cdot \psi) \)

ON THE OTHER HAND:

\( \Pi(\psi) \rightarrow M_\nu \Pi(\psi) \)

\( Z(\psi) \rightarrow \int e^{-S_m} Z(\psi) \)

So

\[ Z(M_\nu \psi) = \int e^{-S_m} Z(\psi) \]
Working perturbatively in $g^2$, choice of contour does not enter. Expanding about the saddle point:

Inverse propagator: $\Delta_{ij} = (\tau + \tau' D)_{ij}$, $M = (A\ B)\ C\ D)$

Vertices: $\Delta_{ij} \in \mathcal{F}_g(W)$,

Where: $\tau_{ij} = \partial_{x^i} p_j$

$\Delta_{ij}$, $\Delta_{ij}^2 = \delta_{ij}$
Genus Two:

\[ \mathcal{F}_2(M_{g,4}) = \mathcal{F}_2(\psi) + \delta^2(\frac{1}{2} \varphi \sigma_i \varphi_j + \frac{1}{2} \varphi \delta \varphi, \varphi_i) + \ldots \]

\[ M_{g,4} \quad \psi \]

Corresponding to all possible (stable) degenerations of genus \( g \) Riemann surface.
Non-trivial monodromy around a singular point corresponds to choosing $A$-cycles that are not well defined:

$x \rightarrow C \cdot p + d \cdot x$

$\det C \neq 0$

Conversely, near a singular pt, the "good" variables are those with no nontrivial monodromy
There is another choice of polarization available:

Pick a background complex structure $\Omega$ on $Y$.

Then, any $\omega \in \mathcal{H}^3(Y, \mathbb{C})$

\[ \omega = \Psi \Omega + z^i \bar{D}i \Omega + \bar{z}^i \bar{D}_i \Omega + \Phi \Omega \]

covariant with respect to Kahler connection: $A_i = 2K$, $\bar{A}_i = \bar{\partial}_i K$

use $z^i, \bar{z}^i$ as coordinates on $\mathcal{H}^3(Y)$
\[ \hat{Z}(z; \phi) = \langle z; \phi | Z \rangle \]

Does not require a choice of symplectic basis of A- and B-cycles.

Correspondingly,

\[ \hat{Z}(z; \phi) \]

is monodromy invariant, and well-defined all over the moduli space. However, it depends on the choice of background \( \Omega \), and not holomorphically.

Moreover, \( \hat{Z} \) and \( Z \) are closely related.
\[ \hat{Z}(z, y) = \int \frac{e^{-S(z, y, x)}}{a_0^2} Z(x) \]

**Wave function**

**Symplectic polarization**

**Expanding about saddle point**

**At \( \omega = \Omega \):**

\[ \hat{F}_2(x, \bar{x}) = F_2(x) + \left( \frac{1}{2\pi} \right)^{\frac{3}{2}} \left( \frac{1}{2} \partial_x \bar{F}_1 + \frac{1}{2} \partial_{\bar{x}} \bar{F}_1 \right) \]

\[ + \left( \frac{1}{2\pi} \right)^{\frac{3}{2}} \left( \frac{1}{2\pi} \right)^{\frac{1}{2}} \left( \frac{1}{2} \partial_x \bar{F}_1 \partial_{\bar{K}} F_0 \right) + \frac{1}{8} \partial_{\bar{K}} F_0 \]

\[ + \left( \frac{1}{2\pi} \right)^{\frac{3}{2}} \left( \frac{1}{2\pi} \right)^{\frac{1}{2}} \left( \frac{1}{2\pi} \right)^{\frac{1}{2}} \left( \frac{1}{2} \partial_{\bar{K}} F_0 \partial_{\bar{L}} F_0 \right) \]

\[ + \frac{1}{12} \partial_{\bar{K}} \partial_{\bar{L}} F_0 \]

**Diagram:**

[Diagram showing the expansion of \( \hat{F}_2 \) with various terms and their contributions]
Thus, have a trade-off.

The price of
holomorphicity of $f_g$

is that they fail to be
precisely modular.

The price of modularity
of $\hat{f}_g$ is that they fail to
be holomorphic!

In either case, the anomaly comes
from boundaries of moduli spaces
of Riemann surfaces.
To summarize:

**Free energy in holomorphic polarization**

i) $\hat{F}_g(x, \bar{x})$ is invariant under \( \Pi \)

ii) $\hat{F}_g(x, \bar{x})$ is "almost" holomorphic

\[ = \text{finite power series in } (z - \bar{z})^{-1} \]

**Free energy in real polarization**

iii) $F_g(x)$ is holomorphic, but not modular

iv) $F_g(x)$ is the constant part of the series expansion of $\hat{F}_g(x, \bar{x})$ in $(z - \bar{z})^{-1}$
Forms of this type were studied by Kaneko and Zagier.

Forms satisfying i) and iii) (with arbitrary weight)

= almost holomorphic modular forms of $\Gamma$.

Moreover, for every almost holomorphic modular form can define the associated quasi-modular form of $\Gamma$, satisfying iii) and iv).

$$f_g = \lim_{\tau \to \infty} \hat{f}_g$$
One can make the symmetry manifest as follows. Note that:

\[ M_0: \quad \frac{1}{\tau - \frac{1}{\bar{\tau}}} \rightarrow (\tau + D) \frac{1}{\tau - \frac{1}{\bar{\tau}}} (\tau + D)^T - C(\tau + D)^T \]

\[ M_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n, \mathbb{Z}) \]

Find \( E^{2g}(\tau) \), such that

\[ \hat{E}^{2g}(\tau, \bar{\tau}) = E^{2g}(\tau) + \left( \frac{1}{\tau - \frac{1}{\bar{\tau}}} \right) \]

transforms as

\[ \hat{E}(\tau, \bar{\tau}) \rightarrow (\tau + D) \hat{E}(\tau + D)^T \]

\( \left( \begin{array}{c} E^{2g}(\tau) \\ E_2(\tau) \end{array} \right) \sim \text{Second Eisenstein series } E_2(\tau) \text{ of } \text{SL}(2, \mathbb{Z}) \)
THAN ONE CAN SHOW

THAT WHILE $\hat{f}_g(x,\bar{x})$ IS NOT
HOLOMORPHIC, AND $f_g(x)$ NOT
MODULAR, AT EACH GENUS, $\exists$ COMBINATIONS
OF THEM ARE BOTH MODULAR
AND HOLOMORPHIC.

$$h_2(x) = F_2(x) - E^{30} \left( \frac{1}{2} e_{1212} \bar{F}_1 + \frac{1}{2} e_{123} \bar{F}_2 \bar{F}_3 \right)$$

$$+ \ldots$$

$$= \hat{F}_2(x, \bar{x}) - \hat{E}^{30} \left( \frac{1}{2} e_{1212} \bar{F}_1 + \frac{1}{2} e_{123} \bar{F}_2 \bar{F}_3 \right)$$

$$+ \ldots$$

$$h_2(x) = \overbrace{\begin{array}{c}
\begin{array}{c}
\text{WEIGHT ZERO} \\
\text{MODULAR}
\end{array}
\end{array}}^{\uparrow}
\begin{array}{c}
\begin{array}{c}
\text{FORM OF } \Gamma
\end{array}
\end{array}
$$
Turning this around,

\[ \hat{f}_g(x) \quad \hat{f}_g(x_{1\ldots n}) \]

are fixed recursively in terms of lower genus amplitudes, up to a weight zero modular form of \( \Gamma \), \( h_g(x) \).

\[ h_g = \text{meromorphic function on the moduli space} \]
THIS (RE)-DERIVES THE SOLUTIONS OF HOLOMORPHIC ANOMALY EQUATIONS OF BCOV USING SYMMETRY ALONE.

THIS WAS ALSO ANTICIPATED BY DIJKGRAAF ('93.)
For non-compact CY manifolds

Based on Riemann surfaces of genus $g$.

$$Y: \quad uv = H(y, z) \in \mathbb{C}^g$$

$$H(y, z) = 0$$

$\Gamma$ is the modular group of the Riemann surface

$g > 0 \Rightarrow$ Siegel (almost) modular forms

$\Gamma \in \text{Sp}(2g, \mathbb{Z})$
\( \mathcal{N}=2, \, d=4 \) GAUGE THEORIES

\[ \mathcal{H}_{SU(N)}: \quad \gamma^2 - R^2(x, u^i) + \Lambda^{2n} = 0 \]

SEIBERG-WITTNEN CURVE

\[ \alpha_1 \quad \ldots \quad \alpha_{n-1} \quad \text{genus} = N-1 \]

TOPOLOGICAL STRING COMPUTES

\[ \int d^4x d^8 \theta \, F_g \, \mathcal{N}^{2g} \]

GRAVIPHOTON SUPERFIELD

\[ \langle a^i | Z \rangle = \exp(\sum_{g} F_g(a^i) \, q_s^{2g^2}) \]

IS A WAVE FUNCTION,
AND \( F_g(a) \) ARE (ALMOST) FIXED
BY MONODROMIES.
(CHECKED, FOR SU(2), USING RESULTS OF NEKRASOV).
• Riemann surfaces appear as mirrors of local toric Calabi-Yau manifolds.
Example:

A-model on local $\mathbb{P}^2$:

$$X = \mathcal{O}(-3) \rightarrow \mathbb{P}^2$$

The mirror is an elliptic curve

$$Y: \quad x_1^3 + x_2^3 + x_3^3 - 3u^2 x_1 x_2 x_3 = x_3^3 u^5$$

With 1-form

$$\omega = \log \left( \frac{x_2}{x_3} \right) \frac{dx_1}{x_1}$$
The moduli space of complex structures has 3 singular points.

\[ \mathbb{C}^3 / \mathbb{Z}_3 \text{ orbifold point} \quad \Psi = 0 \]

\[ \text{Conifold pt} \quad \Psi = 1 \]

\[ \Psi = \infty \quad \text{large radius} \]

\[ \begin{pmatrix} t_D \\ t \end{pmatrix} \sim \begin{pmatrix} 4 \text{ brane} \\ 2 \text{ brane} \end{pmatrix} \]

Around which the periods \( \begin{pmatrix} t_D \\ t \end{pmatrix} \) undergo monodromy.

\[ M_0 = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \]

\[ M_0^3 = 1 \]

\[ M_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

Generates \( \Gamma = \Gamma_0(3) \subset \text{SL}(2, \mathbb{Z}) \)
The large radius basis \((t_D, t)\) non-trivial monodromy around orbifold point \(\psi = 0\).

Good variables correspond to twisted sectors:

\[
\begin{pmatrix}
  s_D \\
  c
\end{pmatrix}
= \begin{pmatrix}
  h^{22} & \text{twisted sector} \\
  h^{11} & \text{twisted sector}
\end{pmatrix}
\]

At orbifold pt.

Which diagonalize the monodromy

\[
M_0 : \begin{pmatrix}
  s_D \\
  c
\end{pmatrix} \rightarrow \begin{pmatrix}
  \alpha^2 s_D \\
  \alpha c
\end{pmatrix}.
\]

\[
\begin{pmatrix}
  t_D \\
  t
\end{pmatrix} = \tilde{M} \begin{pmatrix}
  s_D \\
  c
\end{pmatrix}
\]
ORBIFOLD $\overset{\text{ORB}}{F_g(G)} \sim F_g(t)$ \overset{\text{LARGE RADIUS}}{\longleftrightarrow}

ARE RELATED, NOT BY

ANALYTIC CONTINUATION,

BUT BY FEYNMAN GRAPH EXPANSION ($g > 0$).
PREDICTIONS FOR

GROMOV WITTEN - INVARIANTS

OF $\mathbb{C}^3/\mathbb{Z}_3$, FROM LARGE RADIUS

RESULTS (TOPOLOGICAL VERTEX).

BRYAN ET AL.
YUAN ET AL.

$$F_g^\text{orb}(G) = \sum_{n=1}^{\infty} \frac{N_n^g}{(3n)!} g^{3n}$$

$$N_n^g = \langle O_0 O_0 \cdots O_0 \rangle_g$$

3n

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<th>$g$</th>
<th>$n = 1$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
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<td>$\frac{16571968616003676641}{2^9 \cdot 3^{18} \cdot 5 \cdot 11}$</td>
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</table>
Compact Calabi-Yau

\[ \Gamma_C \cong \text{Sp}(2b_2 + 2, \mathbb{Z}) \]

A new class of modular forms arises since

\[ (\tau - \bar{\tau})_{\text{even}} \]

is not positive definite, but has signature

\[ (1, b_2) \]

New theory of "Lorentzian" modular forms
The B-model has larger symmetry group.

Quantum symmetry of B-model:

ΓC \ w\text{-preserving diffeomorphisms} \\
\overset{\uparrow}{(3,0) \text{ form on CY}}

For local CY = 0 W∞ algebra which fixes the amplitudes

For compact CY = 0 ?