Introduction to Stochastic Analysis

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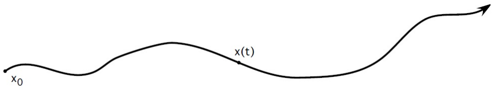
Background and Motivation

Consider the ordinary differential equation:

$$\begin{cases} X'(t) = \mu(X(t), t) \\ X(0) = x_0, \end{cases}$$

which defines a trajectory $x : [0, \infty) \to \mathbb{R}^n$.



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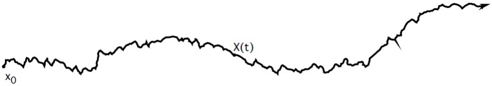

In practice, solutions often display noise. We may want to model in the form:

$$\frac{dX}{dt} = \mu\left(X(t), t\right) + \sigma\left(X(t), t\right) \cdot \eta_t,$$

with η_t satisfying, at least approximately,

- η_{t_1} and η_{t_2} are independent when $t_1 \neq t_2$,
- $\{\eta_t\}$ is stationary, i.e. distribution is translation invariant,
- $\mathbb{E}\left[\eta_t\right] = 0$ for all t.

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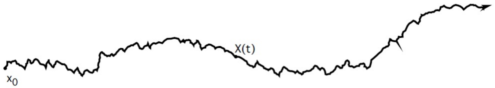

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- $\mathbb{E}\left[\eta_t\right] = 0$ for all t.

It turns out no reasonable stochastic process exists satisfying these.

Background and Motivation

Re-interpret as an integral equation:

$$X(t) = X(0) + \int_0^t \mu\left(X(s), s\right)\, ds + \int_0^t \sigma\left(X(s), s\right)\, dW_s.$$

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Goals of this talk:

- Motivate a definition of the stochastic integral,
- Explore the properties of Brownian motion,
- Highlight major applications of stochastic analysis to PDE and control theory.

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References:

- "An Intro. to Stochastic Differential Equations", L.C. Evans
- "Brownian Motion and Stoch. Calculus", Karatzas and Shreve

Table of contents

Probability Spaces

We want to define a probability space $(\Omega, \mathcal{F}, P)$ to capture the formal notions:

- Ω is a set of "outcomes"
- $\mathcal{F}$ is a collection of "events"
- P measures the likelihood of different "events".

Probability Spaces

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- $\mathcal{F}$ is a collection of "events"
- P measures the likelihood of different "events".

Definition (σ-algebra)

If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a collection $\mathcal{F}$ of subsets on Ω with the following properties:

1. $\emptyset \in \mathcal{F}$
2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$
3. $A_1, A_2, \ldots \in \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Probability Spaces

Definition (Probability measure)

Given a pair $(\Omega, \mathcal{F})$*, then a probability measure* P *is a function* $P : \mathcal{F} \to [0, 1]$ *such that:*

1. $P(\emptyset) = 0, \; P(\Omega) = 1$
2. *If* $A_1, A_2, \ldots \in \mathcal{F}$ *are pairwise disjoint, then*

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

We call a triple $(\Omega, \mathcal{F}, P)$ a probability space.

Random Variables

Definition (Random variable)

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}^n$ is called a random variable if for each $B \in \mathcal{B}$, we have

$$X^{-1}(B) \in \mathcal{F}.$$

Equivalently, we say X is $\mathcal{F}$-measurable.

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Proposition

Let $X : \Omega \to \mathbb{R}^n$ be a random variable. Then

$$\sigma(X) = \{X^{-1}(B) | B \in \mathcal{B}\}$$

is a σ-algebra, called the σ-algebra generated by X.

Random Variables

Proposition

Let $X : \Omega \to \mathbb{R}^n$ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then

$$\mu_X(B) = P\left[X^{-1}(B)\right]$$

for each $B \in \mathcal{B}$ is a measure on $\mathbb{R}^n$ called the distribution of X.

Random Variables

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Let $X : \Omega \to \mathbb{R}^n$ *be a random variable on a probability space* $(\Omega, \mathcal{F}, P)$. *Then*

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for each $B \in \mathcal{B}$ *is a measure on* $\mathbb{R}^n$ *called the distribution of* X.

Definition (Expectation)

Let $X : \Omega \to \mathbb{R}^n$ *be a random variable and* $f : \mathbb{R}^n \to \mathbb{R}$ *be Borel measurable. Then the expectation of* $f(X)$ *may be defined as:*

$$\mathbb{E}\left[f(X)\right] = \int_{\mathbb{R}^n} f(x)\, d\mu_X(x).$$

Random Variables

Definition (Conditional Expectation)

Let X be $\mathcal{F}$-measurable and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Then a conditional expectation of X given $\mathcal{G}$ is any $\mathcal{G}$-measurable function $\mathbb{E}\left[X|\mathcal{G}\right]$ such that

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right] 1_A\right] = \mathbb{E}\left[X\, 1_A\right]$$

for any $A \in \mathcal{G}$.

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for any $A \in \mathcal{G}$.

Proposition (Some Properties of Conditional Expectation)

- *Linearity in $\mathbb{E}[\cdot \mid \mathcal{G}]$.*
- *If X is $\mathcal{G}$-measurable, then $\mathbb{E}[XY \mid \mathcal{G}] = X\,\mathbb{E}[Y \mid \mathcal{G}]$ a.s. as long as XY is integrable.*
- *If $\mathcal{H} \subset \mathcal{G}$, then $\mathbb{E}[X \mid \mathcal{H}] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}]$ a.s.*

Stochastic Processes

Definition (Stochastic process)

- *A collection* $\{X_t : t \in T\}$ *of random variables is called a stochastic process.*
- *For each* $\omega \in \Omega$*, the mapping* $t \mapsto X(t, \omega)$ *is the corresponding sample path.*

Examples:

- Simple random walk
- Markov chain
- $\cdots$

Stochastic Processes

Proposition

Let X_n be a stochastic process. The sequence of σ-algebras defined by:

$$\mathcal{F}_n = \sigma\left(X_0, X_1, \ldots, X_n\right).$$

is an increasing sequence. We call such an increasing sequence of σ-algebras a filtration.

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Definition (Martingale)

Let $\{X_n\}$ be a stochastic process such that each X_n is $\mathcal{F}_n$-measurable. We say X_n is a martingale if

- $X_n \in L^1(\Omega, \mathcal{F}, P)$ *for all* $n \geq 0$
- $\mathbb{E}\left[X_{n+1} | \mathcal{F}_n\right] = X_n$ *for all* $n \geq 0$.

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Proposition

The simple random walk is a martingale if and only if $p = \frac{1}{2}$.

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Proof.

Recall, a simple random walk is $X_n = \sum_{k=1}^{n} \xi_k$, where $\{\xi_n\}_{n\geq 0}$ are IID with $P[\xi_n = 1] = 1 - P[\xi = -1] = p \in (0, 1)$.

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$$\mathbb{E}[|X_n|] \leq \sum_{k=1}^{n} \mathbb{E}[|\xi_k|] = n$$

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$$\begin{aligned} \mathbb{E}[|X_n|] &\leq \sum_{k=1}^{n} \mathbb{E}[|\xi_k|] = n \\ \mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_n + \xi_{n+1}|\mathcal{F}_n] \end{aligned}$$

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\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_n + \xi_{n+1}|\mathcal{F}_n] \\
&= X_n + \mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = X_n + 2p - 1.
\end{aligned}$$

□

Stochastic Processes

Definition (Discrete-time stochastic integration)

Let $\{X_n\}_{n\geq 0}$ and $\{A_n\}_{n\geq 0}$ be two stochastic processes. We define the (discrete-time) stochastic integral of A with respect to X as the process:

$$I_n = \sum_{k=1}^{n} A_k \left(X_k - X_{k-1} \right).$$

Example: Betting strategy...

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If $\{X_n\}$ is a martingale and $\{A_n\}$ is a "predictable", $L^\infty(\Omega)$ process, then I_n is a martingale.

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Proof.

Note, each I_n is $\mathcal{F}_n$-measurable. Holder's inequality shows $I_n \in L^1(\Omega)$. We check the last condition using predictability of A:

$$\begin{aligned} \mathbb{E}\left[I_{n+1}|\mathcal{F}_n\right] &= \mathbb{E}\left[I_n + A_{n+1}\left(X_{n+1} - X_n\right)|\mathcal{F}_n\right] \\ &= I_n + A_{n+1}\mathbb{E}\left[X_{n+1} - X_n|\mathcal{F}_n\right] = I_n. \end{aligned}$$

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Interpretation: Impossible to make money betting on a martingale.

Brownian Motion

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Definition (Brownian Motion)

A real-valued stochastic process W is called a Brownian motion if:

1. $W_0 = 0$ *almost surely,*
2. $W_t - W_s$ *is* $N(0, t - s)$ *for all* $t \geq s \geq 0$,
3. *for all times* $0 < t_1 < t_2 < \cdots < t_n$, *the random variables* $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ *are independent.*

Brownian Motion

Theorem (Sketch of Existence)

Let $\{w_k\}$ be an orthonormal basis on $L^2(0,1)$. Let $\{\xi_k\}$ be a sequence of independent, $N(0,1)$ random variables. The sum

$$W_t(\omega) = \sum_{k=0}^{\infty} \xi_k(\omega) \int_0^t w_k(s)\, ds$$

converges uniformly in t almost surely. W_t is a Brownian motion for $0 \leq t \leq 1$, and furthermore, $t \mapsto W_t(\omega)$ is continuous almost surely.

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Proof

We ignore all technical issues of convergence and just check the joint distributions of increments.

Brownian Motion

A sum of normal random variables is normal.

$$\mathbb{E}\left[W_{t_{m+1}} - W_{t_m}\right] = \sum_{k=1}^{\infty} \int_{t_m}^{t_{m+1}} w_k \, ds \, \mathbb{E}\left[\xi_k\right] = 0.$$

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$$\mathbb{E}\left[\Delta W_{t_m} \Delta W_{t_n}\right] = \sum_{k,l} \int_{t_m}^{t_{m+1}} w_k \, ds \int_{t_n}^{t_{n+1}} w_l \, ds \mathbb{E}\left[\xi_k \xi_l\right]$$

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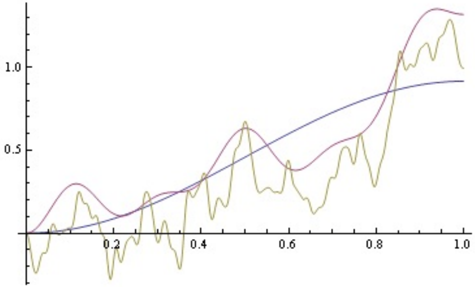
$$\begin{aligned} \mathbb{E}\left[\Delta W_{t_m} \Delta W_{t_n}\right] &= \sum_{k,l} \int_{t_m}^{t_{m+1}} w_k \, ds \int_{t_n}^{t_{n+1}} w_l \, ds \mathbb{E}\left[\xi_k \xi_l\right] \\ &= \sum_{k=0}^{\infty} \int_{t_m}^{t_{m+1}} w_k \, ds \int_{t_n}^{t_{n+1}} w_k \, ds \\ &= \int_0^1 1_{[t_m, t_{m+1}]} 1_{[t_n, t_{n+1}]} \, ds = \Delta t_m \, \delta_n^m. \end{aligned}$$

Brownian Motion

```
Ψ[t_, n_] := Sqrt[2] (Cos[n π t] - 1) / (n π);
c = RandomVariate[NormalDistribution[0, 1], 100];
Plot[{c[[1]] Ψ[t, 1],
  Sum[c[[k]] Ψ[t, k], {k, 1, 10}],
  Sum[c[[k]] Ψ[t, k], {k, 1, 100}]}, {t, 0, 1}]
```



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We would like to develop a theory of stochastic differential equations of the form:

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We interpret this equation in integral form:

$$X(t) = X_0 + \int_0^t \mu(X,s)\,ds + \int_0^t \sigma(X,s)\,dW_s$$

and attempt to define the integral on the right-hand-side.

Stochastic Integration

Definition (Step Process)

A stochastic process $\{A_t\}_{t\in[0,T]}$ *is called a step process if there exists a partition* $0 = t_0 < t_1 < \cdots < t_n = T$ *such that*

$$A_t \equiv A_k \text{ for } t_k \leq t < t_{k+1}.$$

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Definition (Ito Integral for Step Processes)

Let $\{A_t\}_{t\in[0,T]}$ *be a step process, as above. We define an Ito stochastic integral of* A *as*

$$\int_0^T A\,dW_t = \sum_{k=0}^{n-1} A_k \left(W_{t_{k+1}} - W_{t_k}\right).$$

Stochastic Integration

Proposition (Approximation by Step Processes)

Let $A \in L^2(\Omega; L^2(0,T))$. Then there exists a sequence of bounded step processes A_n converging to A in $L^2(\Omega; L^2(0,T))$. Furthermore, we have convergence

$$\int_0^T A_n \, dW_t \overset{L^2(\Omega)}{\to} \int_0^T A \, dW_t.$$

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Remark: There are myriad measurability issues we are glossing over. Typically, we ask that $A : \Omega \times [0,T] \to \mathbb{R}$ is:

- Square-integrable
- "Progressively measurable"
- "Adapted" + continuous, or "predictable"

In this case, the Ito integral of A is a martingale.

Ito's Formula

How do we compute Ito integrals in practice? Ito's formula.

Ito's Formula

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Theorem (Ito's Formula)

Suppose that X_t is a stochastic process satisfying the SDE

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t,$$

for "nice" μ and σ. Let $f : \mathbb{R} \times [0, T] \to \mathbb{R}$ be C^2. Set $Y_t = f(X_t, t)$. Then Y_t satisfies the SDE

$$\begin{aligned} dY_t &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2 \right) dt + \frac{\partial f}{\partial x}\sigma\, dW_t \\ &= \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\, dt. \end{aligned}$$

Ito's Formula

Lemma

1. $d\left(W_t^2\right) = dt + 2W_t\, dW_t$
2. $d\left(tW_t\right) = W_t\, dt + t\, dW_t$

Ito's Formula

Lemma

1. $d\left(W_t^2\right) = dt + 2W_t\, dW_t$
2. $d\left(tW_t\right) = W_t\, dt + t\, dW_t$

Proof.

Let $0 = t_0 < t_1 < \cdots < t_n = t$. Approximate the Ito integral:

$$\sum_{k=0}^{n-1} 2W_{t_k}\left(W_{t_{k+1}} - W_{t_k}\right) = W_{t_n}^2 - \sum_{k=0}^{n}\left(W_{t_{k+1}} - W_{t_k}\right)^2 \xrightarrow{P} W_t^2 - t.$$

Similar for (2).

□

Ito's Formula

Lemma (Ito Product Rule)

Let X_t and Y_t satisfy:

$$\begin{cases} dX_t = \mu_1 \, dt + \sigma_1 \, dW_t \\ dY_t = \mu_2 \, dt + \sigma_2 \, dW_t. \end{cases}$$

Then

$$d\left(X_t Y_t\right) = Y_t \, dX_t + X_t \, dY_t + \sigma_1 \sigma_2 \, dt.$$

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Then

$$d\left(X_t Y_t\right) = Y_t \, dX_t + X_t \, dY_t + \sigma_1 \sigma_2 \, dt .$$

Proof.

Approximate by step processes. Use previous lemma. Be careful about convergence. □

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$$dY_t = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\, dt.$$

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$$dY_t \;=\; \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2\,dt.$$

Proof.

Apply lemmas inductively to compute $d(t^n X_t^m)$. Approximate f, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial f}{\partial t}$ by polynomials. Be careful about convergence. □

Martingale Representation Theorem

Theorem

Let W_t be a Brownian motion with filtration $\mathcal{F}_t$. Let M_t be a continuous, square-integrable martingale with respect to $\mathcal{F}_t$, along with a few other technical, but reasonable, conditions. Then there exists a predictable process ϕ_t such that:

$$M_t = M_0 + \int_0^t \phi_s \, dW_s.$$

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Significance:

- Brownian motion is the archetypal continuous, square-integrable martingale.

Feynman-Kac Formula

Theorem

Consider the parabolic PDE on $\mathbb{R} \times [0, T]$*:*

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} - V(x,t)u + f(x,t) = 0,$$

with the terminal condition $u(x,T) = \Psi(x)$*. Then:*

$$\begin{aligned} u(x,t) &= \mathbb{E}\left[\int_t^T e^{-\int_t^r V(X_\tau,\tau)\,d\tau} f(X_r,r)\,dr \right. \\ &\quad \left. + e^{-\int_t^T V(X_\tau,\tau)\,d\tau}\Psi(X_T) \mid X_t = x\right] \end{aligned}$$

where X_t *is a solution to the SDE*

$$dX_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t.$$

Feynman-Kac Formula

Proof.

Define a stochastic process:

$$Y_r = e^{-\int_t^r V(X_\tau,\tau)\,d\tau} u(X_r, r) + \int_t^r e^{-\int_t^s V(X_\tau,\tau)\,d\tau} f(X_s, s)\,ds.$$

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Apply Ito's formula, use the PDE to cancel a lot of terms, and get:

$$Y_T = Y_t + \int_t^T e^{-\int_t^s V(X_\tau,\tau)\,d\tau} \sigma(X_s, s) \frac{\partial u}{\partial x}\,dW_s.$$

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Taking conditional expectations on each side and using the martingale-property of Ito integrals, we get:

$$u(x,t) = \mathbb{E}\left[Y_t \mid X_t = x\right] = \mathbb{E}\left[Y_T \mid X_t = x\right].$$

Hamilton-Jacobi-Bellman Equation

Consider a process X which is driven by a control α_t via the SDE:

$$dX_t = \mu(X_t, \alpha_t, t)\,dt + \sigma(X_t, \alpha_t, t)\,dW_t.$$

Consider the optimization problem:

$$V(x,t) = \max_{\alpha(\cdot)} \left\{ \mathbb{E}\left[\int_t^T r(X_s, \alpha_s, s)\,ds + g(X_T) \mid X_t = x \right] \right\}.$$

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Theorem (HJB Equation)

Assuming μ, σ, r, and g are all "nice", V is a solution (in a weak sense) to the fully non-linear PDE:

$$\begin{cases} 0 = \frac{\partial V}{\partial t} + \sup_\alpha \left\{ r(x,\alpha,t) + \mu(x,\alpha,t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(x,\alpha,t)^2 \frac{\partial^2 V}{\partial x^2} \right\} \\ V(x,T) = g(x) \end{cases}$$

Hamilton-Jacobi-Bellman Equation

Proof

To illustrate the main idea, we proceed formally, assuming that a optimal control α_t^* exists and everything in sight is smooth.

Hamilton-Jacobi-Bellman Equation

Proof

To illustrate the main idea, we proceed formally, assuming that a optimal control α_t^* exists and everything in sight is smooth. For any $\epsilon > 0$, for sufficiently small $0 < h < T - t$, we have for all $\alpha \in \mathbb{R}$:

$$V(x,t) + \epsilon \geq \mathbb{E}\left[\int_t^{t+h} r(X_s, \alpha, s)\, ds + V(X_{t+h}, t+h) \mid X_t = x\right]$$

$$V(x,t) - \epsilon \leq \mathbb{E}\left[\int_t^{t+h} r(X_s, \alpha_t^*, s)\, ds + V(X_{t+h}, t+h) \mid X_t = x\right].$$

Hamilton-Jacobi-Bellman Equation

Then, we conclude:

$$2\epsilon \geq |V(x,t) - \sup_{\alpha} \left\{ \mathbb{E}\left[\int_t^{t+h} r(X_s, \alpha, s)\, ds + V(X_{t+h}, t+h) \mid X_t = x \right] \right\}$$

Hamilton-Jacobi-Bellman Equation

Then, we conclude:

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Applying Ito's formula to the term inside the expectation, we see:

$$\begin{aligned} & \int_t^{t+h} r(X_s, \alpha, s)\, ds + V(X_{t+h}, t+h) \\ = \quad & V(X_t, t) + \int_t^{t+h} \left(r + \frac{\partial V}{\partial t} + \mu \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} \right) ds \\ & + \int_t^{t+h} \cdots \, dW_s. \end{aligned}$$

Hamilton-Jacobi-Bellman Equation

Now, using the martingale property of Ito integrals, we obtain:

$$2\epsilon \geq \left|\sup_{\alpha}\left\{\mathbb{E}\left[\int_t^{t+h}\left(r+\frac{\partial V}{\partial t}+\mu\frac{\partial V}{\partial x}+\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}\right)\,ds \mid X_t = x\right]\right\}\right|.$$

Hamilton-Jacobi-Bellman Equation

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Then taking $\epsilon, h \to 0$, we obtain:

$$0 = \frac{\partial V}{\partial t} + \sup_{\alpha}\left\{r(x,\alpha,t) + \mu(x,\alpha,t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(x,\alpha,t)^2\frac{\partial^2 V}{\partial x^2}\right\}.$$

□

Conclusion

Stochastic analysis provides a mathematical framework for uncertainty quantification and local descriptions of global stochastic phenomena.

- Deep connections to elliptic and parabolic PDE,
- Characterizes large classes of continuous-time stochastic processes,
- Fundamental tools for the working analyst.