Duality Methods in Portfolio Allocation with Transaction Constraints and Uncertainty

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Project Overview

We examine optimal portfolio allocation with transaction constraints via duality methods

- Various models of asset returns and transaction costs
- Correlations and uncertainty in estimated parameters
- General algorithm to solve dual portfolio allocation problem.

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Rapidly approximate optimal allocations with a large number of assets and uncertain parameters.

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Problem Description

Single-period investment model with n assets:

- Risk-return preference: $f = f_1 + \cdots + f_n$,
- Transaction costs: $g = g_1 + \cdots + g_n$,
- Investment constraint: $w_i \in [\underline{w}_i, \overline{w}_i] = \mathcal{W}_i$.

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Assume that we can rapidly optimize:

$$\min_{w_i \in \mathcal{W}_i} f_i(w_i) + \lambda g_i(w_i).$$

Lagrangian Relaxation

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In this project, we consider the related problem:

$$d^* = \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \mathcal{L}(w, \lambda)$$

where $\mathcal{L}(w, \lambda) = f(w) + \lambda (g(w) - \tau)$.

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Proposition

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Let $w^* \in \mathcal{W}$ such that $g(w^*) \leq \tau$ and $f(w^*) = p^*$.

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Let $w^* \in \mathcal{W}$ such that $g(w^*) \leq \tau$ and $f(w^*) = p^*$. For all $\lambda \geq 0$,

$$\begin{aligned} p^* &\geq f(w^*) + \lambda\left(g(w^*) - \tau\right) \\ &\geq \min_{w \in \mathcal{W}} \left\{f(w) + \lambda\left(g(w) - \tau\right)\right\}. \end{aligned}$$

Lagrangian Relaxation

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With the dual problem defined as above, $p^* \geq d^*$.

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Then

$$p^* \geq \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \mathcal{L}(w, \lambda) = d^*.$$

Binary Model

In the binary model, we have fixed transaction costs for purchases and disallow short-sales:

$$g_i(\xi) = \begin{cases} +\infty & \text{if } \xi < 0 \\ 0 & \text{if } \xi = 0 \\ b_i & \text{if } \xi > 0. \end{cases}$$

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Theorem

We can construct an optimal solution (λ^*, w^*) to the dual problem in the binary model in time $O(n)$.

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Corollary

There is no duality gap in the binary model with unit transaction costs and integer-valued τ.

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Proof

$$d^* = \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \{f(w) + \lambda (g(w) - \tau)\}$$

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Proof

$$\begin{aligned} d^* &= \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \{f(w) + \lambda (g(w) - \tau)\} \\ &= \max_{\lambda \geq 0} \left\{ -\lambda\tau + \sum_{i=1}^{n} \min_{w_i \in \mathcal{W}_i} f_i(w_i) + \lambda g_i(w_i) \right\} \end{aligned}$$

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$$
\begin{aligned}
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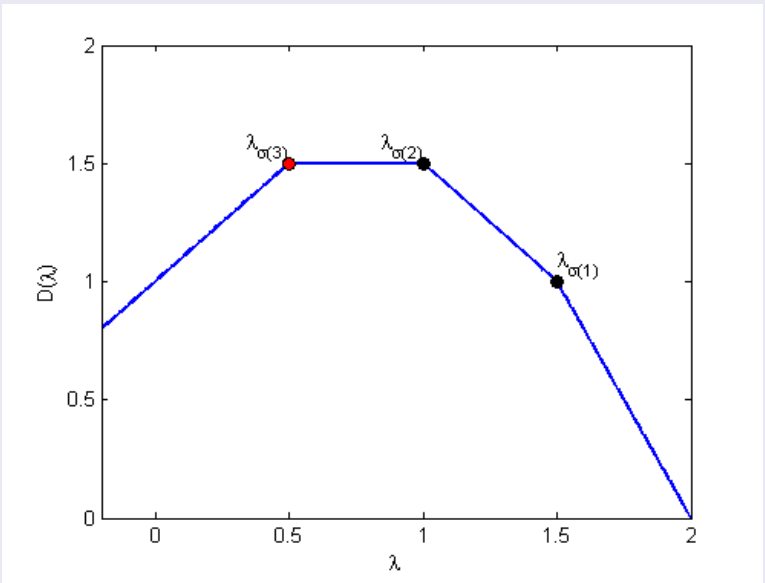
&= \max_{\lambda \geq 0} \left\{ -\lambda\tau + \sum_{i=1}^{n} \min \left\{ f_i(0), \lambda b_i + \min_{0 < w_i \leq \overline{w}_i} f_i(w_i) \right\} \right\}.
\end{aligned}
$$

Binary Model

$$d^* = \max_{\lambda \geq 0} \left\{ -\lambda\tau + \sum_{i=1}^{n} \min\left\{ f_i^0, \lambda b_i + f_i^+ \right\} \right\}.$$



Maximization in $O(n)$ via Quickselect algorithm. □

Ternary Model

In the ternary model, we have fixed transaction costs for both purchases and sales:

$$g_i(\xi) = \begin{cases} s_i & \text{if } \xi < 0 \\ 0 & \text{if } \xi = 0 \\ b_i & \text{if } \xi > 0. \end{cases}$$

Furthermore, let us assume that we have the restriction on τ that

$$0 \leq \tau \leq \sum_{i=1}^{n} \max(s_i, b_i).$$

Ternary Model

Proposition

The solution of the dual problem under the ternary model may be written

$$d^* = 1^\top f^0 + \min_{u^\pm} \Big\{ \left(h^+\right)^\top u^+ + \left(h^-\right)^\top u^- : u^\pm \geq 0, \\ u^+ + u^- \leq 1, \\ b^\top u^+ + s^\top u^- \leq \tau \Big\}$$

for appropriate vectors f^0, h^+, and h^-.

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Theorem

We can construct an optimal solution (λ^*, w^*) to the dual problem in the ternary model in polynomial time.

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In the ternary model, it is not true that $p^* = d^*$ in general.

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Proposition

Let (λ^*, w^*) be an optimal solution to the dual problem obtained from the algorithm above. Then the duality gap is bounded by

$$0 \leq p^* - d^* \leq \lambda^* \left(\tau - g(w^*)\right).$$

Correlation Model

In the correlation model, we consider an objective function:

$$f(w, \hat{r}) = \frac{1}{2} w^\top \Sigma w - \hat{r}^\top w,$$

where $\hat{r} \in \mathcal{R} \subset \mathbb{R}^n$ contains predicted returns and Σ is an estimate of the covariance matrix.

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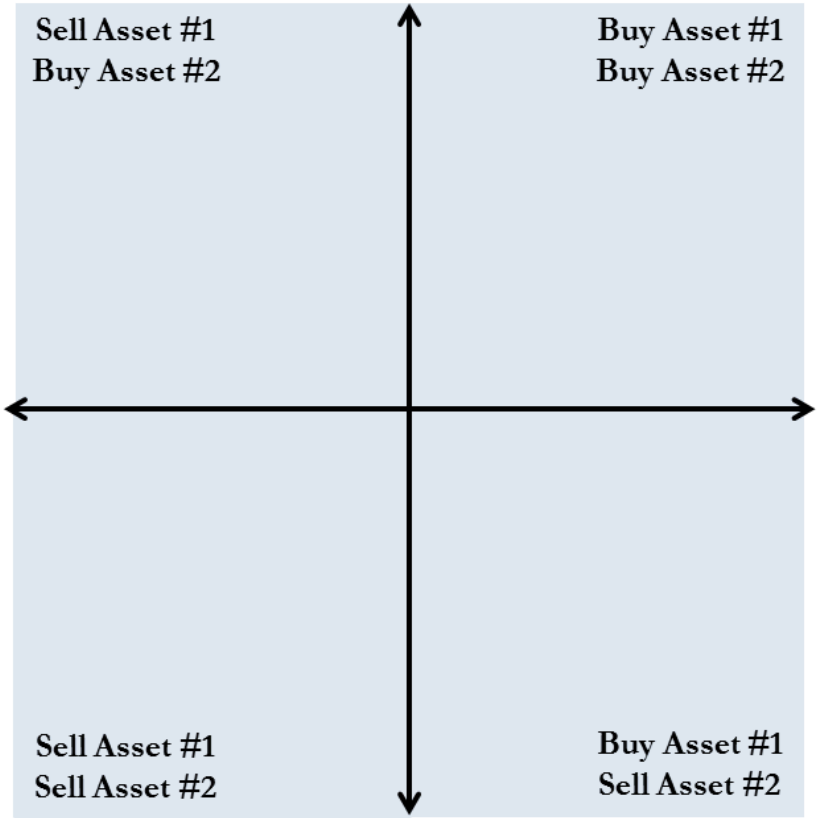
The corresponding dual problem is:

$$d^* = \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \max_{\hat{r} \in \mathcal{R}} \{ f(w, \hat{r}) + \lambda (g(w) - \tau) \}$$

Branch-and-Bound Method



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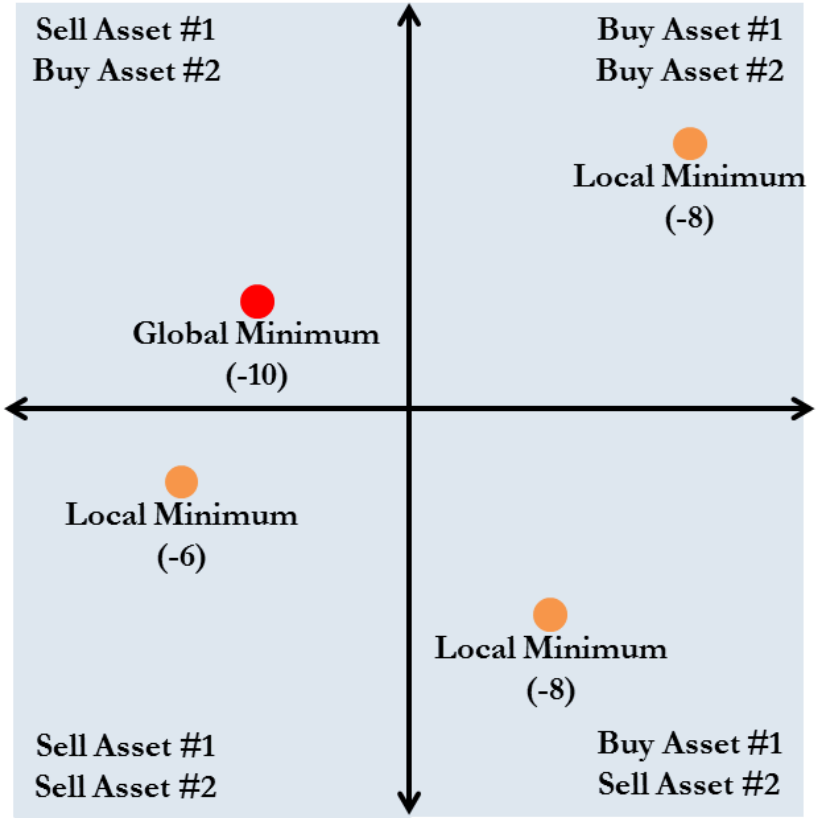

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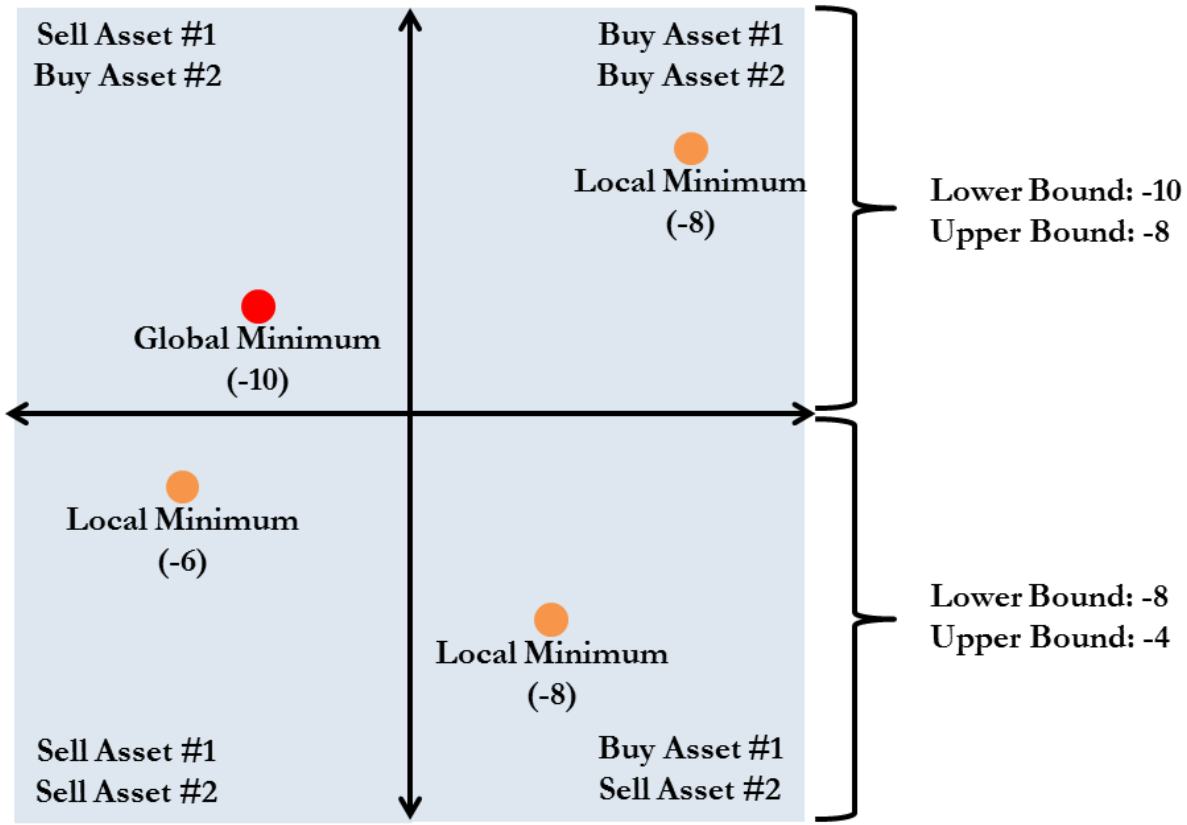

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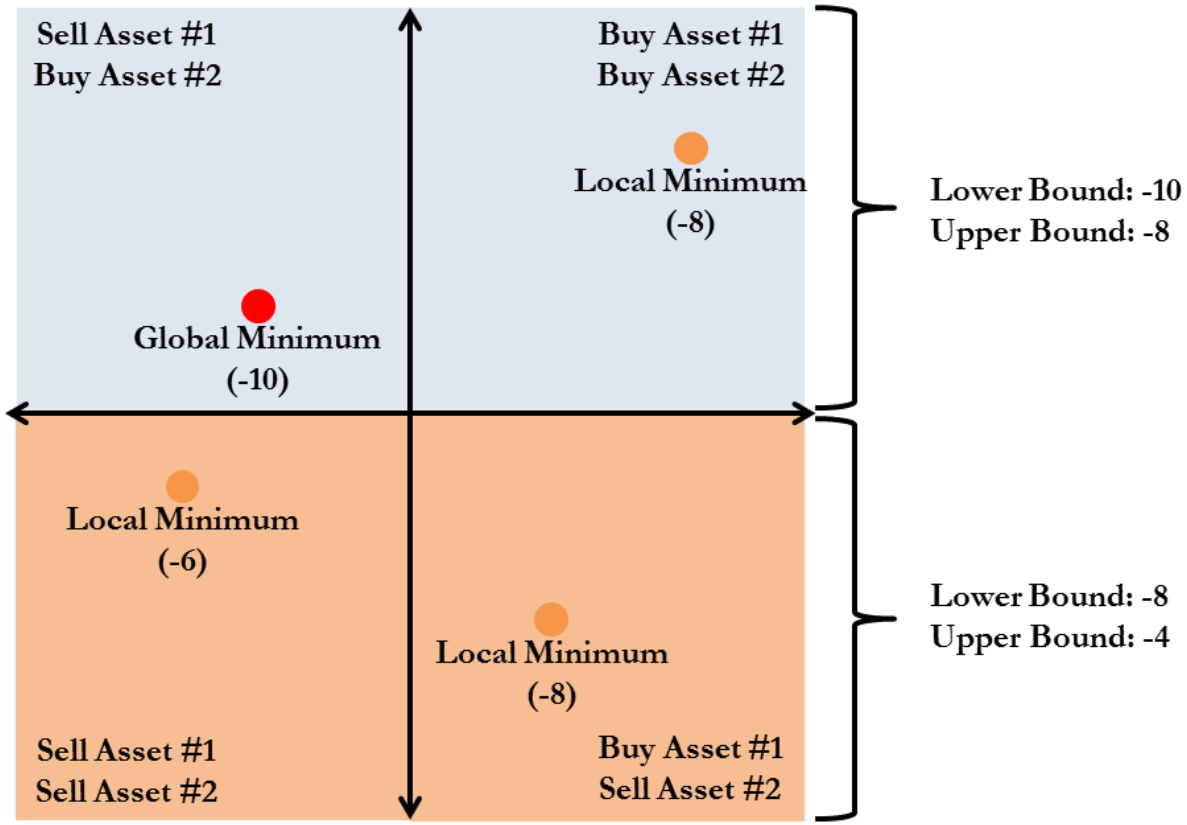
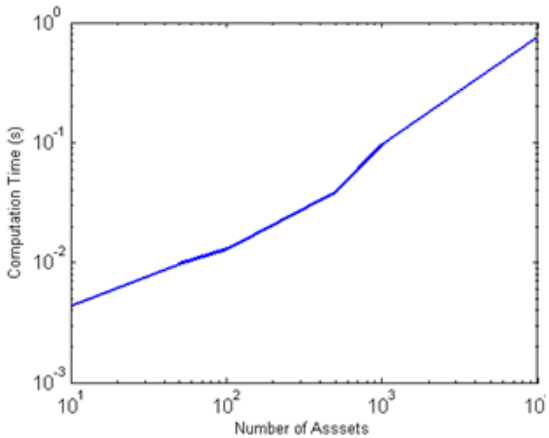
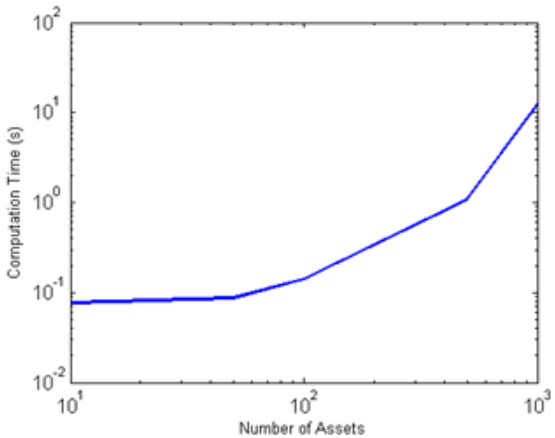


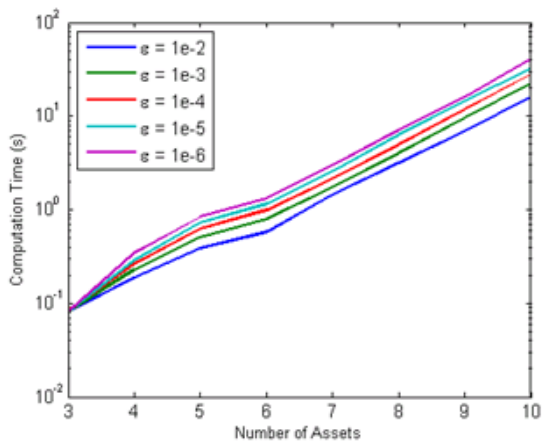
Numerical Tests



Binary Model



Ternary Model



Correlation Model

Tested under extra condition that $\sum_{i=1}^{n} w_i \leq 1$.

Conclusions

Duality methods provide approximate allocations in difficult portfolio optimization problems:

- Usefulness of decomposability
- Opportunities for parallelization in branch-and-bound methods
- Approximate solution as input to solver for the primal problem.