Duality Methods in Portfolio Allocation with Transaction Constraints and Uncertainty

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We examine optimal portfolio allocation with transaction constraints via duality methods

- Various models of asset returns and transaction costs
- Correlations and uncertainty in estimated parameters
- General algorithm to solve dual portfolio allocation problem.
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- Various models of asset returns and transaction costs
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Rapidly approximate optimal allocations with a large number of assets and uncertain parameters.
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Problem Description

Single-period investment model with $n$ assets:

- Risk-return preference: $f = f_1 + \cdots + f_n$,
- Transaction costs: $g = g_1 + \cdots + g_n$,
- Investment constraint: $w_i \in [\underline{w}_i, \overline{w}_i] = \mathcal{W}_i$. 

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Duality Methods in Portfolio Allocation
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Assume that we can rapidly optimize:

$$\min_{w_i \in \mathcal{W}_i} f_i(w_i) + \lambda g_i(w_i).$$
In general, we consider an investment problem of the form:

\[ p^* = \min_{w \in \mathcal{W}} \{ f(w) : g(w) \leq \tau \}. \]
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\]

In this project, we consider the related problem:

\[
d^* = \max_{\lambda \geq 0} \min_{w \in W} \mathcal{L}(w, \lambda)
\]

where \( \mathcal{L}(w, \lambda) = f(w) + \lambda (g(w) - \tau) \).
Proposition

With the dual problem defined as above, \( p^* \geq d^* \).
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Proof.

Let \( w^* \in \mathcal{W} \) such that \( g(w^*) \leq \tau \) and \( f(w^*) = p^* \).
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\[
\begin{align*}
p^* & \geq f(w^*) + \lambda (g(w^*) - \tau) \\
& \geq \min_{w \in \mathcal{W}} \{ f(w) + \lambda (g(w) - \tau) \}.
\end{align*}
\]
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p^* \geq f(w^*) + \lambda (g(w^*) - \tau)
\geq \min_{w \in \mathcal{W}} \{ f(w) + \lambda (g(w) - \tau) \}.
\]

Then

\[
p^* \geq \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \mathcal{L}(w, \lambda) = d^*.
\]
In the binary model, we have fixed transaction costs for purchases and disallow short-sales:

\[ g_i(\xi) = \begin{cases} +\infty & \text{if } \xi < 0 \\ 0 & \text{if } \xi = 0 \\ b_i & \text{if } \xi > 0. \end{cases} \]

Theorem
We can construct an optimal solution \((\lambda^*, \omega^*)\) to the dual problem in the binary model in time \(O(n)\).

Corollary
There is no duality gap in the binary model with unit transaction costs and integer-valued \(\tau\).
In the binary model, we have fixed transaction costs for purchases and disallow short-sales:

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d^* = \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \{ f(w) + \lambda (g(w) - \tau) \}
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d^* = \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \left\{ f(w) + \lambda (g(w) - \tau) \right\}
\]
\[
= \max_{\lambda \geq 0} \left\{ -\lambda \tau + \sum_{i=1}^{n} \min_{w_i \in \mathcal{W}_i} \left\{ f_i(w_i) + \lambda g_i(w_i) \right\} \right\}
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**Theorem**

We can construct an optimal solution \((\lambda^*, w^*)\) to the dual problem in the binary model in time \(O(n)\).

**Proof**

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\]

\[
= \max_{\lambda \geq 0} \left\{ -\lambda \tau + \sum_{i=1}^{n} \min_{0 \leq w_i \leq \overline{w}_i} \left\{ f_i(0), \lambda b_i + \min_{0 < w_i \leq \overline{w}_i} f_i(w_i) \right\} \right\}.
\]
Binary Model

\[ d^* = \max_{\lambda \geq 0} \left\{ -\lambda \tau + \sum_{i=1}^{n} \min \left\{ f_i^0, \lambda b_i + f_i^+ \right\} \right\}. \]

Maximization in \( O(n) \) via Quickselect algorithm.
In the ternary model, we have fixed transaction costs for both purchases and sales:

$$g_i(\xi) = \begin{cases} 
  s_i & \text{if } \xi < 0 \\
  0 & \text{if } \xi = 0 \\
  b_i & \text{if } \xi > 0.
\end{cases}$$

Furthermore, let us assume that we have the restriction on $\tau$ that

$$0 \leq \tau \leq \sum_{i=1}^{n} \max(s_i, b_i).$$
The solution of the dual problem under the ternary model may be written

\[ d^\star = 1^\top f^0 + \min_{u^\pm} \left\{ (h^+)^\top u^+ + (h^-)^\top u^- : u^\pm \geq 0, \\
                                 u^+ + u^- \leq 1, \\
                                 b^\top u^+ + s^\top u^- \leq \tau \right\} \]

for appropriate vectors \( f^0, h^+, \) and \( h^- \).
Proposition

The solution of the dual problem under the ternary model may be written

$$d^* = 1^T f^0 + \min_{u\pm} \left\{ (h^+)^T u^+ + (h^-)^T u^- : u^\pm \geq 0, u^+ + u^- \leq 1, b^T u^+ + s^T u^- \leq \tau \right\}$$

for appropriate vectors $f^0$, $h^+$, and $h^-$. 

Theorem

We can construct an optimal solution $(\lambda^*, w^*)$ to the dual problem in the ternary model in polynomial time.
Proposition

In the ternary model, it is not true that $p^* = d^*$ in general.
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Proposition

Let $(\lambda^*, w^*)$ be an optimal solution to the dual problem obtained from the algorithm above. Then the duality gap is bounded by

$$0 \leq p^* - d^* \leq \lambda^* (\tau - g(w^*))$$.
In the correlation model, we consider an objective function:

\[ f(w, \hat{r}) = \frac{1}{2} w^\top \Sigma w - \hat{r}^\top w, \]

where \( \hat{r} \in \mathcal{R} \subset \mathbb{R}^n \) contains predicted returns and \( \Sigma \) is an estimate of the covariance matrix.
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where \( \hat{r} \in \mathcal{R} \subset \mathbb{R}^n \) contains predicted returns and \( \Sigma \) is an estimate of the covariance matrix.

The corresponding dual problem is:

\[ d^* = \max_{\lambda \geq 0} \min_{w \in \mathcal{W}} \max_{\hat{r} \in \mathcal{R}} \{ f(w, \hat{r}) + \lambda (g(w) - \tau) \} \]
Branch-and-Bound Method

```
Sell Asset #1
Buy Asset #2

Buy Asset #1
Buy Asset #2

Sell Asset #1
Sell Asset #2
```
Branch-and-Bound Method

- Sell Asset #1
- Buy Asset #2
- Buy Asset #1
- Sell Asset #2

Global Minimum (-10)
Local Minimum (-6)
Local Minimum (-8)
Local Minimum (-8)
Branch-and-Bound Method

- Sell Asset #1
- Buy Asset #2
- Buy Asset #1
- Buy Asset #2

Global Minimum (-10)
Local Minimum (-6)
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Lower Bound: -10
Upper Bound: -8

Lower Bound: -8
Upper Bound: -4
Branch-and-Bound Method

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- Buy Asset #2
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- Local Minimum (-8)
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- Sell Asset #2
- Lower Bound: -10
- Upper Bound: -8
- Lower Bound: -8
- Upper Bound: -4

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Tested under extra condition that $\sum_{i=1}^{n} w_i \leq 1$. 
Conclusions

Duality methods provide approximate allocations in difficult portfolio optimization problems:

- Usefulness of decomposability
- Opportunities for parallelization in branch-and-bound methods
- Approximate solution as input to solver for the primal problem.