Proposition: Lex is a monomial ordering on $\mathbb{N}^n = \mathbb{Z}^n_{\geq 0}$.

<u>Proof (scetch):</u>

- Total ordering since we compare entry wise and $\mathbb{N}$ is totally ordered

- If $\alpha >_{lex} \beta \Rightarrow$ left most non-zero entry of $\alpha - \beta \Rightarrow$ say $\alpha_i - \beta_i > 0$

$$\alpha + \gamma > \beta + \gamma \quad \forall \gamma$$

$$(\alpha+\gamma) - (\beta+\gamma) = \alpha - \beta$$

$$\therefore \quad \alpha_i - \beta_i > 0$$

- well ordering also follows from $\mathbb{N}$ being well ordered

Suppose

$\alpha(1) >_{lex} \alpha(2) >_{lex} \cdots$ was infinitly decreasing.

Since we entry wise the sequence must stabalize in $\mathbb{N}$ $\therefore$ there could be no such sequence.

Note: in Lex $x > y > z$

$$x >_{lex} y^5 z^3 \quad , \quad y^2 > y z^{277}$$

<u>Graded Lex Order</u> = grlex

$\alpha, \beta \in \mathbb{N}^n$

$\alpha >_{grlex} \beta$ if

$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n}$ OR $|\alpha| = |\beta|$ and $\alpha >_{lex} \beta$

[· check total degree
· Break tie with lex]

$(2, 2, 3) <_{grlex} (1, 3, 4)$

$x^2 y^2 z^3 \qquad x y^3 z^4$

$(2, 1, 3) <_{grlex} (2, 2, 2)$

$x^2 y z^3 \qquad x^2 y^2 z^2$

Since $(2,2,2) - (2,1,3) = (0, \underline{1}, -1)$.

Graded Reverse Lex order = grevlex

$\alpha, \beta \in \mathbb{N}^n$

$\alpha > \beta$ if

$|\alpha| = \sum \alpha_i > |\beta| = \sum \beta_i$ OR $|\alpha| = |\beta|$ and right most nonzero entry is <u>negative</u>.

$(2, 4, 3) >_{grevlex} (1, 5, 3)$

$x^2 y^4 z^3 \qquad x y^5 z^3$

$(2,4,3) - (1,5,3) = (1, -1, 0)$

biggest in grlex　　biggest grevlex.

$$g = 5x^3 + \underbrace{7x^2z^2}_{} + \underbrace{4xy^2z}_{} + 4z^2$$

decreasing lex order

Def's

$f = \sum_{\alpha} a_\alpha x^\alpha \in K[x_1, \dots, x_n]$, > a monomial order

- $\text{multidegree}(f) = \max(\alpha \in \mathbb{N}^n \mid a_\alpha \neq 0)$

  in lex

  $\text{multidegree}(g) = (3, 0, 0)$

- Leading coefficent

  $LC(f) = a_{\text{multideg}(f)} \in K$

  lex

  $LC(g) = -5$

- Leading monomial: $LM(f) = x^{\text{multidegree}(f)} = (x_1 \cdots x_n)^{\text{multidegree}(f)}$

  lex

  $LM(g) = x^3$

- Leading term $= LC(f) \cdot LM(f)$

  in lex

  i.e. $LT(g) = -5x^3$

Lemma

$f, g \neq 0 \in K[x_1, \dots, x_n]$

- $\text{multidegree}(f \cdot g) = \text{multidegree}(f) + \text{multidegree}(g)$

- $f + g \neq 0$ then

$$\text{multidegree}(f+g) \leq \max(\text{multidegree}(f), \text{multideg}(g))$$

$=$ if $\text{multideg}(f) \neq \text{multideg}(g)$

$f = x^2 + yz$　　$g = -x^2 + y^2 + z$　　$f+g =$

$\text{multideg}(f) = \text{multideg}(g) = (2,0,0)$.

$\text{multideg}(f+g) = (0,2,0)$

Division Alg. in $k[x_1, \dots, x_n]$

For $f, f_1, \dots, f_s \in k[x_1, \dots, x_n]$ we want to divide f by $f_1, \dots, f_s$ i.e. we wish to find

"quotients"

remainder

$$f = q_1 f_1 + \dots + q_s f_s + r$$

Ex) Divide $f = x^2y + xy^2 + y^2$ by $f_1 = xy - 1$, $f_2 = y^2 - 1$

- use $>$ = lex order
- work with leading terms

$q_1: \dfrac{LT(f)}{LT(f_1)} = \dfrac{x^2y}{xy} = x$

$q_2:$

$$\begin{array}{l|l}
f_1 = xy - 1 & x^2y + xy^2 + y^2 = f \\
f_2 = y^2 - 1 & \underline{x^2y - x} \quad = f_1 \cdot \dfrac{LT(f)}{LT(f_1)} \\
 & xy^2 + x + y^2 = \hat{f}
\end{array}$$

$LT(f_1) \mid LT(\hat{f})$ so use this

$$\hat{q}_1 : x + \frac{LT(\hat{f})}{LT(f_1)} = x + y, \qquad \frac{LT(\hat{f})}{LT(f_1)} = \frac{xy^2}{xy} = y$$

$\hat{q}_2:$

$$xy^2 + x + y^2 = \hat{f}$$
$$- \left(f_1 \cdot \frac{LT(\hat{f})}{LT(f_1)} \right)$$

$$xy^2 + x + y^2 = \hat{f}$$
$$- (xy^2 - y)$$
$$\overline{\quad x + y^2 + y = p \quad}$$

$LT(f_1) = xy$, $LT(f_2) = y^2$, do not divide $LT(p) = x$

$\therefore$ put $LT(p)$ in r ← remainder

r: x

proceded

$$\hat{p} = y^2 + y$$