Elimination

We can do this with a Groebner Basis

Def: Given $I = (f_1, \dots, f_s) \subseteq k[x_1, \dots, x_n]$ the ℓ^{th} elimination ideal in $k[x_{\ell+1}, \dots, x_n]$ is

$$I_\ell = I \cap k[x_{\ell+1}, \dots, x_n].$$

Theorem (The Elimination Theorem)

Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal and let G be a Gröbner basis of I w.r.t lex $x_1 > \dots > x_n$. Then for every $0 \le \ell \le n$

$$G_\ell = G \cap k[x_{\ell+1}, \dots, x_n]$$

is a Gröbner basis of I_ℓ ← ℓ^{th} elim. ideal.

Proof: Fix ℓ. $G_\ell \subseteq I_\ell$ by construction, so need to check $(LT(I_\ell)) = (LT(G_\ell))$

Certainly $(LT(G_\ell)) \subseteq (LT(I_\ell))$. Show other direction.

Let $f \in I_\ell$ Show $LT(f)$ is divisible by $LT(g)$ for some $g \in G_\ell$

$f \in I_\ell \Rightarrow f \in I \therefore LT(g) \mid LT(f)$ for some $g \in G$ since G is a GB of I.

$f \in I_\ell \Rightarrow LT(f)$ only has var. $x_{\ell+1}, \dots, x_n$ appearing

$\Rightarrow LT(g)$ only has var. $x_{\ell+1}, \dots, x_n$ appearing

$LT(g) \in k[x_{\ell+1}, \dots, x_n]$

we are using $\underline{\text{Lex}}$ $x_1 > \dots > x_n$ $\therefore$ any monomia

in $x_1, \dots, x_\ell$ is $\underline{\text{greater}}$ than $\underline{\text{all}}$ monomials in

$K[x_{\ell+1}, \dots, x_n]$

$\therefore$ since $LT(g) \in K[x_{\ell+1}, \dots, x_n] \Rightarrow g \in K[x_{\ell+1}, \dots, x_n]$

$\therefore g \in G_\ell$ $\therefore LT(f) \in (LT(G_\ell))$.

Ex) $I_2 = I \cap \mathbb{C}[z] = (\underbrace{z^6 - 4z^4 + 4z^3 - z^2}_{g(z)})$

solutions for $g(z)$ are called partial solutions for $V(I)$

$I_1 = I \cap \mathbb{C}[y,z] = (z^6 - 4z^4 + 4z^3 - z^2,\ 2yz^2 + z^4 - z^2,\ y^2 - y - z^2 + z)$

$I_0 = I$

Our procedure of solving I_2, then using for I_1 etc

is called "Exension"

- To describe points in $V(I)$ work one coord at a time $\Rightarrow$ at the ℓ^{th} step we have $(a_{\ell+1}, \dots, a_n) \in V(I_\ell)$ is a partial solution. Now want to find

a_ℓ s.t. $(a_\ell, a_{\ell+1}, \dots, a_n) \in V(I_{\ell-1})$

i.e. If $I_{\ell-1} = (g_1, \dots, g_r) \leq K[x_\ell, \dots, x_n]$

want $x_\ell = a_\ell$ which solve

$$g_1(x_\ell, a_{\ell+1}, \dots, a_n) = \dots = g_r(x_\ell, a_{\ell+1}, \dots, a_n) = 0$$

Problem this might not have a solution.

i.e. want to find roots of $\gcd(g_1(x_\ell), \dots, g_r(x_\ell))$

Ex) $I = (xy-1,\ xz-1)$

$I = (y-z,\ xz-1)$ ← G.B.

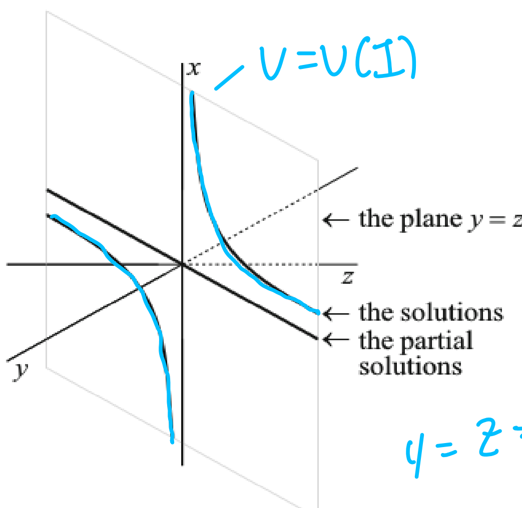
$I_1 = I \cap \mathbb{C}[y,z] = (y-z)$

$(y,z) = (a,a)$ $\quad a \in \mathbb{R}$ is a partial solution.

$$x = \frac{1}{z}$$

this extends to a complete solution

$\left(\frac{1}{a}, a, a\right)$ $\quad a \in \mathbb{R}$, but not for $a = 0$



$y = z = 0$ is not in $V(I)$

Focus on the case where we eliminate ~~just~~ the first variable

Q: can $(a_2, \dots, a_n) \in V(I_1)$ be extended to a solution $(a_1, \dots, a_n) \in V(I)$?

A:

this is needed:

Theorem (Extension Thm.) Let $I = (f_1, \dots, f_s) \leq \mathbb{C}[x_1, \dots, x_n]$ and let I_1 be the first elimination ideal.

For each $1 \leq i \leq s$ write

$$f_i = c_i(x_2, \dots, x_n)\, x_1^{N_i} + \text{terms with deg in } x_1 < N_i.$$

$(N_i \geq 0)$ $c_i \neq 0 \in \mathbb{C}[x_2, \dots, x_n]$. Suppose $(a_2, \dots, a_n) \in V(I_1)$

If $(a_2, \dots, a_n) \notin V(c_1, \dots, c_s)$ then there exists $a_1 \in \mathbb{C}$ s.t. $(a_1, a_2, \dots, a_n) \in V(I)$.

Proof: Section 3.5 (and 3.6)

Geometrically = we are "lifting a projection"

why we need $\mathbb{C}$

$I = (x^2 - y, x^2 - z)$ Eliminating x

we get $y = z \Rightarrow (a, a) \in V(I_1)$ is a partial solution

the L.C. of $x^2 - y$, $x^2 - z$ are 1

coefficients of x^2

$\Rightarrow V(1)$ is empty
so all points extend

$\therefore$ Extension theorem say $(a, a) \in V(I_1)$ always extends to $(\sqrt{a}, a, a) \in V(I)$

If $k = \mathbb{R}$ negative a do not extend

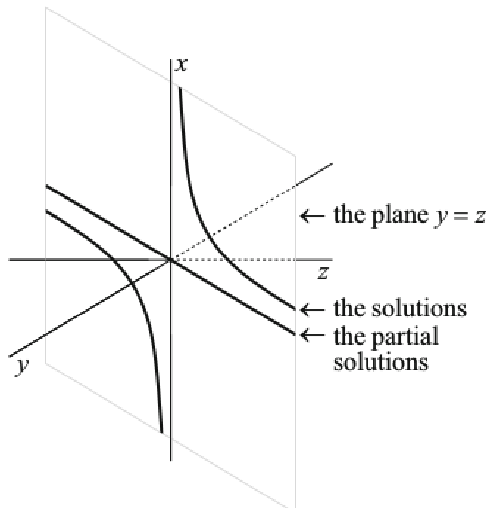
$\therefore$ we need $\mathbb{C}$.

Why we must avoid $V(c_1, \dots, c_s)$

$I = (xy-1,\ xz-1)$

L.C. of x are

$c_1 = y,\ c_2 = z$



$\therefore$ we need to avoid

$V(y,z) = \{(0,0)\}$

Note $V(c_1, \dots, c_s)$ depends on the basis $(f_1, \dots, f_s)$ of I.

In Projective space all solutions extend.