Reduced Groebner Bases are unique:

Lemma: Let G be a GB of $I \subset k[x_1, \dots, x_n]$, $p \in G$ s.t. $LT(p) \in (LT(G \setminus \{p\}))$. Then $G \setminus \{p\}$ is also a GB.

minimal GB: GB G where make $LC = 1$ and remove any redundant $p \in G$.

Def: A <u>reduced</u> Gröbner basis G of I is a GB s.t.

- $LC(p) = 1 \quad \forall p \in G$
- $\forall p \in G$ no monomomial of p lies in $(LT(G \setminus \{p\}))$

This is how M2 checks equality of ideals.

Theorem: Let $I \neq \{0\}$ be a poly ideal. For a fixed monomial order I has a unique reduced Gröbner basis.

Proof: say $g \in G$ is is fully reduced for G if no monomial of g is in $(LT(G \setminus \{g\}))$

To get to reduced GB take a minimal GB

remove p's in a whose LT is already there

$$g' = \overline{g}^{G \setminus \{g\}}$$

$$G' = (G \setminus \{g\}) \cup \{g'\}$$

$LT(g') = LT(g)$ since $LT(g)$ not in $(LT(G \setminus \{g\}))$

$$\therefore (LT(G')) = (LT(G))$$

Do this until all $g' \in G'$ are fully reduced (Gives Existence)

uniqueness: say $G, \tilde{G}$ are reduced GB

$\Rightarrow \quad LT(G) = LT(\hat{G})$ Since minimal

$\therefore \exists \; g \in G, \; \tilde{g} \in \tilde{G}$ s.t $LT(g) = LT(\hat{g})$

Show $g = \hat{g}$. $g - \tilde{g} \in I$.

$\therefore \quad \overline{g - \hat{g}}^{G} = 0$ but $LT(g) = LT(\hat{g})$

$\therefore$ LT's cancel in $g - \hat{g}$

and other terms are divisible by none of $LT(G)$ since reduced

$\therefore \overline{g - \check{g}}^{G} \Rightarrow g - \hat{g} = 0 \quad \blacksquare$

Take away : To check is $I = (f_1, .., f_s)$ and $J = (g_1, .., g_s)$ we compute a reduced GB.

Applications of GB

- Ideal membership $f \in I$ iff $\overline{f}^{G} = 0$
- Ideal equality (above)
- Solving poly eqn's.

Ex) $\quad z = \pm \frac{1}{2}\sqrt{\pm\sqrt{5} - 1}$

Implicitization

Say we have a parametric variety in K^n ← (assume f's poly)

$$x_1 = f_1(t_1, \dots, t_m)$$
$$\vdots$$
$$x_n = f_n(t_1, \dots, t_m)$$

want to find Implicit eqn's in x_i's

Consider an aff var. in K^{m+n}

$$x_1 - f_1(t_1, \dots, t_m) = 0$$
$$\vdots$$
$$x_n - f_n(t_1, \dots, t_m) = 0$$

the idea is to eliminate t_i's

we can do this with GB. in lex in $k[t_1, \dots, t_m, x_1, \dots, x_n]$
$t_1 > \dots > t_m > x_1 > \dots > x_n$

Refinements of Buchberger Criterion

Buchberger ⟺ a basis $G = \{g_1, \dots, g_t\}$ of I is a GB iff $\overline{S(g_i, g_j)}^G = 0 \quad \forall i, j$

i.e. each S-poly

$$S(g_i, g_j) = \sum q_\ell g_\ell \text{ to using div}$$

Def Fix a mono. order let $G = \{g_1, \dots, g_t\} \subset k[x_1, \dots, x_n]$

$f \in k[x_1, \dots, x_n]$ reduces to zero modulo G

$f \to_G 0$ if f has a standard representation

$f = A_1 g_1 + \dots + A_t g_t$, $A_i \in k[x_1, \dots, x_n]$

s.t. when $A_i g_i \neq 0$

$\text{multdeg}(f) \geq \text{multdeg}(A_i g_i)$.