Monomials Ideals + Dickson Lemma

- For mon. ideals we can solve ideal description
  ↑ show all mon. ideals have a finite basis

Def: Let $A \subseteq \mathbb{N}^n$ (possibly infinite)

$I = (x^\alpha \mid \alpha \in A) \subseteq K[x_1, \dots, x_n]$

is a <u>monomial ideal</u>.

Ex) $I = (x^2 z, y^3 z, x^2 y z, z^7) \subseteq K[x,y,z]$

Lemma). $I = (x^\alpha \mid \alpha \in A)$ a monomial ideal

$x^\beta \in I$ iff x^β is divisable by x^α for some $\alpha \in A$.

Proof: $\Leftarrow$ If $x^\alpha \mid x^\beta$ for some $\alpha \in A \Rightarrow x^\beta = x^\gamma \cdot x^\alpha$

$\therefore x^\beta \in I$.

$\Rightarrow x^\beta \in I$

$x^\beta = \sum h_i x^{\alpha(i)}$ $\quad h_i \in K[x_1, \dots, x_n]$, $\alpha(i) \in A$

But x^β is a <u>monomial</u> $\therefore x^\beta = h_i x^{\alpha(i)}$ for some i

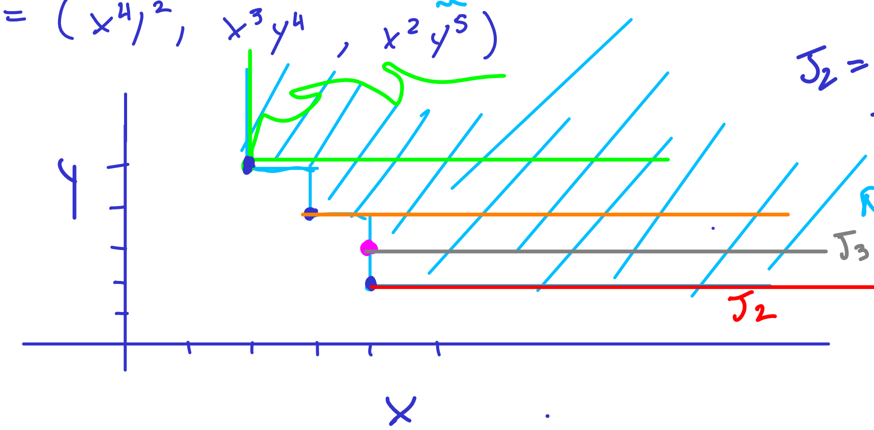
∎

$x^\beta \in I \Leftrightarrow x^\beta = x^\gamma x^\alpha$ for some $\gamma \in \mathbb{N}^n$

$\Rightarrow \alpha + \mathbb{N}^n = \{\alpha + \gamma \mid \gamma \in \mathbb{N}^n\}$

$I = (\underbrace{x^2y^5}_{J_5}, \underbrace{x^3y^4}_{\subseteq J_4}, \underbrace{x^4y^3}_{\subseteq J_3}, \underbrace{x^4y^2}_{\subseteq J_2}, \underbrace{0}_{J_1 = J_0})$ ↑ all exponents of monomials in I $\quad m = 5$

Ex) $I = ((x^4)^2, x^3y^4, x^2y^5)$ $\quad m = 5$



$J_2 = J_3 = (x^4)$

$J_4 = (x^3)$

↖ all monomials in I.

$J_0 = J_1 = \{0\}$

Lemma I monomial ideal, $f \in K[x_1, \ldots, x_n]$

TFAE:

- $f \in I$
- Every term of f lies in I
- f is a K-linear combo. of monomials in I

Cor Monomial ideals I, J are the same iff they contain the same monomials.

Dickson's Lemma

A may be infinite

Thm. $I = (x^{\alpha} \mid \alpha \in A) \subseteq K[x_1, \ldots, x_n]$ a monomial ideal

Then $I = (x^{\alpha(1)}, \ldots, x^{\alpha(s)})$ where $\alpha(1), \ldots, \alpha(s) \in A$

i.e. I has a finite basis.

Proof: Induction on $n = \#$ var

$n=1$ $\quad I$ gen by $\{x_1^{\alpha_1} \mid \alpha \in A \subseteq \mathbb{N}\}$

Let β be the least element of $A \Rightarrow x_1^{\beta} \mid x_1^{\alpha} \quad \forall \alpha \in A$

$$I = (x_1^{\beta})$$

Assume true for $n-1$, consider $I \subseteq K[x_1, \ldots, x_{n-1}, \overset{x_n}{\overset{\shortparallel}{y}}]$

J = ideal in $K[x_1, \ldots, x_{n-1}]$ gen. by monomial

x^{α} s.t. $x^{\alpha} y^m \in I$ for some $m \geq 0$

$J = (x^{\alpha(1)}, \ldots, x^{\alpha(s)}) \subseteq k[x_1, \ldots, x_{n-1}]$

By def. $x^{\alpha(i)} y^{m_i} \in I \quad \forall i$ for some $m_i \geq 0$

$m = \max(m_i)$

J_ℓ = ideal in $k[x_1, \ldots, x_{n-1}]$ gen. by x^β s.t. $x^\beta y^\ell \in I$

$J_\ell = (x^{\alpha_\ell(1)}, \ldots, x^{\alpha_\ell(s_\ell)})$

Claim: I is generated by

$J_m = J: \; x^{\alpha(1)} y^m, \ldots, x^{\alpha(s)} y^m$

$J_{m-1}: \; x^{\alpha_{m-1}(1)} y^{m-1}, \ldots, x^{\alpha_{m-1}(s_{m-1})} y^{m-1}$

$\vdots$

$J_0 = x^{\alpha_0(1)}, \ldots, x^{\alpha_0(s_0)}$

Every monomial in I is div. by one of these

Let $x^\alpha y^p \in I$

If $p \geq m \Rightarrow x^\alpha y^p$ is div. by $x^{\alpha(i)} y^m$

in J

If $p \leq m-1$

$x^\alpha y^p$ is div $x^{\alpha_p(j)} y^p \in J_p$

$\therefore$ every mon. in I is div. by one in $J = J_m, \ldots, J_0$

$\therefore$ by Lemma these mon. gen. ideal with the same monomials as I

By cor. $I = (J_{m_1}, \dots, J_{\infty})$

$\uparrow$ mon. ideals are the same iff they have the same mon.

$\therefore$

$$I = (x^{\beta(1)}, \dots, x^{\beta(s)}) \quad \left(\text{write: } x_1, \dots, x_n = x_1, \dots, x_{n-1}, y\right)$$

Need to show

$$I = (x^{\alpha(1)}, \dots, x^{\alpha(s)}), \quad \alpha(i) \in A \quad \left(\text{i.e. } \beta(i) \text{ might not be in } A\right)$$

but $\quad x^{\beta(i)} \in I = (x^{\alpha} \mid \alpha \in A)$

$\therefore \quad x^{\alpha(i)} \mid x^{\beta(i)}$ for some $\alpha(i) \in A$

$\therefore \quad I = (x^{\alpha(1)}, \dots, x^{\alpha(s)}) \quad \alpha(i) \in A.$ $\blacksquare$

Corol Let $>$ be a realation on $\mathbb{N}^n$ s.t

- $>$ is a total ordering
- If $\alpha > \beta$, $\gamma \in \mathbb{N}^n \Rightarrow \alpha + \gamma > \beta + \gamma$

Then $>$ is a well ordering iff $\alpha \geq 0 \quad \forall \alpha \in \mathbb{N}^n$

Proof: $\Rightarrow$ Assume well ordering, say $\alpha_0 \in \mathbb{N}^n$ least ele.

If $0 > \alpha_0 \Rightarrow \alpha_0 > 2\alpha_0$ this is a cont.

$\Leftarrow$ say $\alpha \geq 0 \ \forall \alpha \in \mathbb{N}^n$, $A \subseteq \mathbb{N}^n$ nonempty

$I = (x^\alpha \mid \alpha \in A)$

By Dickson's $I = (x^{\alpha(1)}, \dots, x^{\alpha(s)})$

$\alpha(1) < \cdots < \alpha(s)$

Claim: $\alpha(1)$ is least ele. of A

since if $\alpha \in A \Rightarrow x^\alpha \in I \therefore x^\alpha$ is div by $x^{\alpha(i)}$

$\therefore \alpha = \alpha(i) + \gamma$, $\gamma \geq 0$

$\therefore \alpha \geq \alpha(i)$

$\therefore \alpha(1)$ is least ele.

Prop) A mon. ideal I has a basis $x^{\alpha(1)}, \dots, x^{\alpha(s)}$ s.t. $x^{\alpha(i)} \nmid x^{\alpha(j)} \ \forall i \neq j$
This basis is unique and is called minimal basis of I.

H.B Thm. + G.B.

Leading term Ideal of I = $(LT(I))$

Def) $I \subseteq K[x_1, \dots, x_n]$ an ideal, fix a mon. order.

$\cdot LT(I) = \{ cx^\alpha \mid \exists f \in I \setminus \{0\}$ with $LT(f) = cx^\alpha \}$

set of all leading terms

- $(LT(I))$ - ideal generated by $LT(I)$.