What about rational parametrizations?

map F
$$\begin{cases} x_1 = \dfrac{f_1(t_1,\dots,t_m)}{g_1(t_1,\dots,t_m)} \\ \vdots \\ x_n = \dfrac{f_n(t_1,\dots,t_m)}{g_n(t_1,\dots,t_m)} \end{cases} \qquad f_i, g_i \in k[t_1,\dots,t_m]$$

$F : k^m \to k^n$ may not be defined on all k^m

i.e. in $V(g_1 \cdots g_n)$ — union of $V(g_i)$'s $= \bigcup_i V(g_i)$

We need to fix this....

Let $\boxed{W = V(g_1 \cdots g_n)} \subseteq k^m$

$F(t_1,\dots,t_m) = \left(\frac{f_1}{g_1}, \dots, \frac{f_n}{g_n}\right)$

defines a map $F : k^m \setminus W \to k^n$

Want the <u>smallest</u> variety in k^n containing $F(k^m \setminus W)$.

we add a variable to work in $k[y, t_1,\dots,t_m, x_1,\dots,x_n]$

$g = g_1 \cdots g_n$

$$J = (g_1 x_1 - f_1, \dots, g_n x_n - f_n, 1 - gy)$$

clear denominators

if $g = 0$ $1 - gy = 0$ has no solutions

Geometric Setup

$$j : k^m \setminus W \longrightarrow k^{1+m+n}$$

$$(t_1, \dots, t_m) \mapsto \left(\frac{1}{g(t_1,\dots,t_m)}, t_1, \dots, t_m, \frac{f_1}{g_1}, \dots, \frac{f_n}{g_n} \right)$$

$$\pi_{1+m} : k^{1+m+n} \longrightarrow k^n$$

$$(y, t_1, \dots, t_m, x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$$

giving

$$\begin{tikzcd} & k^{1+m+n} \arrow[dr, "\pi_{1+m}"] & \\ k^m \setminus W \arrow[ur, "j"] \arrow[rr, "F"] & & k^n \end{tikzcd}$$

$$j(k^m \setminus W) = V(J)$$

$$j(k^m \setminus W) \subseteq V(J)$$ (plugging a point $x_i = \frac{f_i}{g_i}$, $y = \frac{1}{g}$)

If $(y, t_1, \dots, t_m, x_1, \dots, x_n) \in V(J)$

then $g(t_1, \dots, t_m)\, y = 1 \Rightarrow$ no g_i's vanish at $(t_1, \dots, t_m)$

$\Rightarrow$ we can solve $g_i(t_1, \dots, t_m)\, x_i = f(t)$

to get $x_i = \frac{f_i}{g_i}$

$$y = \frac{1}{g(t_1, \dots, t_m)}$$

$$(y, t_1, \dots, t_m, x_1, \dots, x_n) \in j(k^m \setminus W)$$

$$\therefore F(k^m \setminus W) = \pi_{1+m}(j(k^m \setminus W)) = \pi_{1+m}(V(J)).$$

!! Now we can do elimination again!

Theorem) k infinite field. $F: k^m \setminus W \to k^n$ — our parametrization $x_i = \frac{f_i}{g_i}$

$J = (g_1 x_1 - f_1, \ldots, g_n x_n - f_n, 1 - gy) \subseteq k[y, t_1, \ldots, t_m, x_1, \ldots, x_n]$

$g = g_1 \cdots g_n$, $W = V(g) \subseteq k^m$

$J_{1+m} = J \cap k[x_1, \ldots, x_n] = (1+m)^{th}$ elimination ideal of J

Then $V(J_{1+m})$ is the smallest variety in k^n containing $F(k^m \setminus W)$.

Proof: we know $F(k^m \setminus W) = \pi_{1+m}(V(J)) \subseteq V(J_{1+m})$ ↑ Lemma 1 3.2

Show that if $h \in k[x_1, \ldots, x_n]$ vanishes on $F(k^m \setminus W) \Rightarrow h \in J_{1+m}$

[This will show that any variety Z containing $F(k^m \setminus W)$ also contains $V(J_{1+m})$]

Let $h \in k[x_1, \ldots, x_n]$, suppose h vanishes on $F(k^m \setminus W)$.

Divide hg^N (for some $N \in \mathbb{N}$) by $g_i x_i - f_i$ $\forall i$

$$g^N h(x_1, \ldots, x_n) = q_1(g_1 x_1 - f_1) + \cdots + q_n(g_n x_n - f_n) + r(t_1, \ldots, t_m)$$

For any $a = (a_1, \dots, a_m) \in k^m \setminus W$

$$g_i(a) \neq 0$$

$\therefore$ can sub $t_i = a_i$, $x_i = \dfrac{f_i(a)}{g_i(a)}$

$$0 = g(a)^N h\left(\frac{f_1(a)}{g_1(a)}, \dots, \frac{f_n(a)}{g_n(a)}\right) = 0 + \dots + 0 + r(a)$$

$$r(a) = 0 \quad \forall a \in k^n \setminus W$$

Since k is infinite $\Rightarrow$ r = zero polynomial

$$\therefore g^N h(x_1, \dots, x_n) = q_1(g_1 x_1 - f_1) + \dots + q_n(g_n x_n - f_n)$$

$$\Rightarrow g^N h \in (g_1 x_1 - f_1, \dots, g_n x_n - f_n) \subseteq k[t_1, \dots, t_m, x_1, \dots, x_n]$$

$$h = g^N y^N h + h(1 - g^N y^N)$$

$$= y^N (g^N h) + h(1 + gy + \dots + g^{N-1} y^{N-1})(1 - gy)$$

in $k[y, t_1, \dots, t_m, x_1, \dots, x_n]$

$$h \in (g_1 x_1 - f_1, \dots, g_n x_n - f_n, 1 - gy) = J$$

and by assumption

$$h \in k[x_1, \dots, x_n]$$

$$\therefore h \in J_{1+m}$$

Since any $Z = V(h_1, \dots, h_s)$ containing $F(k^m \setminus W)$

h_i must on $F(k^m \setminus W)$ $\therefore$ $h_i \in J_{1+m}$ $\therefore$ $V(J_{1+m}) \subseteq Z$ $\blacksquare$

Take Away:

Given a rational parametrization

Eliminating $y, t_1, \ldots, t_m$ from

$$J = (g_1x_1 - f_1, \ldots, g_nx_n - f_n, 1 - gy)$$

gives smallest variety containg image of Parametrization

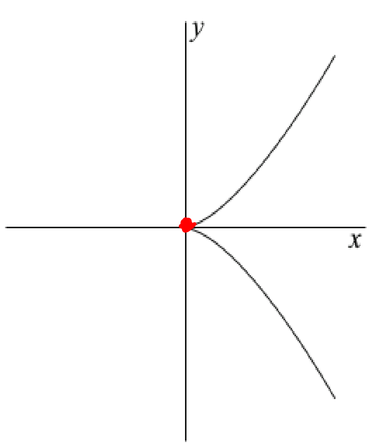
$\therefore$ Solved Implicitization! $\smile$

Alg: • Compute lex $y > t_1 > \cdots > t_m > x_1 > \cdots > x_n$ Gröbner basis of J

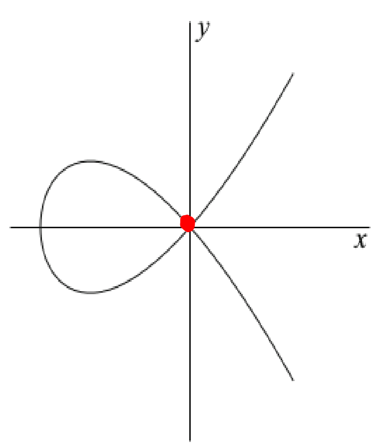
• Elements of GB with <u>only</u> $x_1, \ldots, x_n$ give a GB for smallest var. in k^n containg the para...

Singular Points on a Curve

- Plane curves in k^2 def. by $f(x,y) = 0$, $f \in k[x,y]$
- Expect a well-defined tangent line a most points on curve
- This may fail



$y^2 = x^3$



$y^2 = x^2(1+x)$

want tangent line to be unique and follow curve on both sides of a point

Consider $(a,b) \in V(f)$

L = line through (a,b) = $\begin{aligned} x &= a + ct \\ y &= b + dt \end{aligned}$, $t \in K$

Def $\rfloor$ m positive integer. $(a,b) \in V(f)$, L a line through (a,b)