The Zariski Topology

Def If V is a variety a subset $S \subseteq V$ is *Zariski dense* in V if $V = \overline{S}$.

- closed sets are varieties, i.e. $V(f_1, \dots, f_s) = V(I)$ are closed.

- open sets are formed by complements of closed sets i.e. $k^n \setminus V(I)$

  ↑ Distinguished open sets

- open sets $k^n \setminus V(I)$ are dense in k^n :

- $\pi_\ell(V)$ is Zariski dense in $V(I_\ell)$ (for alg. closed k)

Ex) $V = V(xz, yz)$, $W = V(z)$

↑ union x-y plane and z-axis

↑ x-y plane

$$V \setminus W = z\text{-axis} - (0,0,0)$$

$$\overline{V \setminus W} = z\text{-axis}$$

↑ Can we do this with ideals?

Def: I, J ideals in $K[x_1, \dots, x_n]$

$$I : J = \{ f \in K[x_1, \dots, x_n] \mid f \cdot g \in I \quad \forall g \in J \}$$

Note f may not be in I

ideal quotient or colon ideal

Prop: $I:J$ is an ideal which contains I

- $f \in I:J$, $h \in K[x_1, \dots, x_n] \Rightarrow f \cdot g \in I \quad \forall g \in J$

$h \cdot fg \in I$

since I is an ideal

$\therefore hf \in I:J$

Prop: I, J ideals in $K[x_1, \dots, x_n]$

(i) $V(I) = V(I+J) \cup V(I:J)$

(ii) V, W varieties

$$V = (V \cap W) \cup \overline{(V \setminus W)} \leftarrow$$

(iii) $\overline{V(I) \setminus V(J)} \subseteq V(I:J)$

proof: (ii) $V \setminus W \subseteq V \Rightarrow \overline{V \setminus W} \subseteq V$

and $V \cap W \subseteq V$

$\Rightarrow (V \cap W) \cup \overline{(V \setminus W)} \subseteq V$

$V = V \cap W \cup (V \setminus W) \subseteq V \cap W \cup \overline{(V \setminus W)}$

$V = V \cap W \cup \overline{(V \setminus W)}$

(iii) Show $I:J \subseteq I(V(I)\setminus V(J))$

Let $f \in I:J$, $a \in V(I)\setminus V(J)$

$\Rightarrow fg \in I \quad \forall g \in J, \ a \in V(I)$

$\Rightarrow f(a)g(a) = 0 \quad \forall g \in J$

$a \notin V(J)$

$\therefore \exists g \in J$ s.t. $g(a) \neq 0$

$\Rightarrow f(a) = 0 \quad \forall a \in V(I)\setminus V(J)$

$\therefore f \in I(V(I)\setminus V(J))$

$I:J \subseteq I(V(I)\setminus V(J))$

$V(I:J) \supseteq V(I(V(I)\setminus V(J)))$

$= \overline{V(I)\setminus V(J)}$.

(i) by (ii)

$V(I) = V(I+J) \cup \overline{V(J)\setminus V(J)}$

by (iii) $\subseteq V(I+J) \cup V(I:J)$.

$I \subseteq I+J, \quad I \subseteq I:J$

$V(I+J) \subseteq V(I), \quad V(I:J) \subseteq V(I)$

$\therefore \quad V(I+J) \cup V(I:J) \subseteq V(I)$

So do we have

$$\overline{V(I)\setminus V(J)} \subseteq V(I:J)$$

equal? No :(

not even over $\mathbb{C}$

$$I = (x^2(y-1)), \quad J = (x)$$

$$V(I) = V(x) \cup V(y-1) = V(J) \cup V(y-1)$$

$$\therefore \overline{V(I)\setminus V(J)} = V(y-1).$$

$$(x^2(y-1)):(x) = (x(y-1))$$

$$V(x(y-1)) \neq V(y-1)$$

$$V(I:J^2) = V(y-1) = \overline{V(I)\setminus V(J)}$$

<u>Def:</u> I, J are ideals in $k[x_1,\dots,x_n]$ then the <u>saturation</u> of I w.r.t J is

$$I:J^\infty = \{ f \in k[x_1,\dots,x_n] \mid \forall g \in J, \exists N \geq 0 \text{ s.t. } fg^N \in I \}$$

Prop) I, J ideals in $k[x_1,\dots,x_n]$. $I:J^\infty$ is an ideal. Also:

(i) $I \subseteq I:J \subseteq I:J^\infty$

(ii) $I:J^\infty = I:J^N$ for $N > m$ for some m.

(iii) $\sqrt{I:J^\infty} = \sqrt{I}:J$

Proof: $J^{N+1} \subseteq J^N \Rightarrow I:J^N \subseteq I:J^{N+1}$

less g's to require $f \cdot g \in I$

$\Rightarrow$ $I \subseteq I:J \subseteq I:J^2 \subseteq \dots\dots$

By the ACC $\exists N$ s.t. $I:J^N = I:J^{N+1}$

$\boxed{\text{Show } I:J^\infty = I:J^N}$

- If $f \in I:J^N$, $g \in J \Rightarrow g^N \in J^N$

$\Rightarrow f g^N \in I$ for some $N \Rightarrow f \in I:J^\infty$

$\boxed{I:J^N \subseteq I:J^\infty}$

- $f \in I:J^\infty$, $J = (g_1,\dots,g_s)$

$\Rightarrow f g_i^{m_i} \in I \quad \forall i$

$M = \max(m_i)$

$\Rightarrow f g_i^M \in I \quad \forall i$

$J^{sM} \subseteq (g_1^m, \dots, g_s^m)$

$\Rightarrow f J^{sm} \subseteq I$

$\uparrow$ each term has g_i^m

$\Rightarrow f \in I : J^{sm}$

$I : J^{sm}$ must appear in

in the A.C. $I \subseteq I:J \subseteq I:J^2 \subseteq \cdots$

which stabalizes at N.

$\Rightarrow$ Either $I:J^{sm} \subseteq I:J^N$

$\therefore f \in I:J^N$

(iii) show $\sqrt{I:J^\infty} \supseteq \sqrt{I}:J$

$f \in \sqrt{I:J^\infty} \Rightarrow f^m \in I:J^\infty$

for $g \in G \Rightarrow f^m g^N \in I$ for some N

$\Rightarrow (fg)^{\max(m,N)} \in I \Rightarrow fg \in \sqrt{I} \quad \forall g \in J$

$\therefore f \in \sqrt{I}:J$

Show $\sqrt{I}:J \subseteq \sqrt{I:J^\infty}$

$f \in \sqrt{I}:J$, $J = (g_1, \ldots, g_s)$

$\Rightarrow fg_i \in \sqrt{I}$ $\therefore \exists$ m_i s.t.

$(fg_i)^{m_i} \in I \quad \forall i$

$f^{m_i} g_i^{m_i} \in I \quad \forall i$

$\Rightarrow f^{m_i} \in I : J^{s\hat{m}} \subseteq I : J^{\infty}$

$\Rightarrow f \in \sqrt{I : J^{\infty}}$. $\blacksquare$

Thm) I, J ideals in $k[x_1, \dots, x_n]$

(i) $V(I) = V(I + J) \cup V(I : J^{\infty})$

(ii) $\overline{V(I) \setminus V(J)} \subseteq V(I : J^{\infty})$

(iii) if k alg. closed then

$$V(I : J^{\infty}) = \overline{V(I) \setminus V(J)}$$

Proof: (ii)

Show $I(V(I) \setminus V(J)) \subseteq \sqrt{I} : J$

• $f \in I(V(I) \setminus V(J))$. If $g \in J$

$\Rightarrow f \cdot g$ vanish on $V(I)$

$\therefore fg \in I(V(I)) \Rightarrow fg \in \sqrt{I}$ by Nullstell...

by def: $f \in \sqrt{I} : J$

$\therefore I(V(I) \setminus V(J)) \subseteq \sqrt{I} : J$

Taking "V"

$V(\sqrt{I} : J) \subseteq V(I(V(I) \setminus V(J))) = \overline{V(I) \setminus V(J)}$

$$V(I:J^\infty) = V(\sqrt{I:J^\infty})$$
$$= V(\sqrt{I}:J)$$

$$\therefore \quad V(I:J^\infty) \subseteq \overline{V(I)\setminus V(J)}$$

For k-alg closed

$$V(I) = V(I+J) \cup V(I:J^\infty)$$
$$\Updownarrow$$
$$(V(I)\cap V(J)) \cup \overline{(V(I)\setminus V(J))}$$

Cor) Let I, J, ideals in $k[x_1,\dots,x_n]$ (alg. closed)
and $I = \sqrt{I}$ radical: then

$$V(I:J^\infty) = V(\sqrt{I:J^\infty}) = V(\sqrt{I}:J)$$
$$= V(I:J) = \overline{V(I)\setminus V(J)}$$

$W, V \subseteq k^n$ arbitrary, k arbitrary

$$I(V):I(W) = I(V\setminus W)$$

Thm) I ideal, $g \in k[x_1,\dots,x_n]$. Then
(1) If $I \cap (g) = (h_1,\dots,h_p) \Rightarrow I:g = \left(\frac{h_1}{g},\dots,\frac{h_p}{g}\right)$

(ii) $I = (f_1, \dots, f_s)$, $\hat{I} = (f_1, \dots, f_s, 1 - yg) \subseteq k[x_1, \dots, x_n, y]$

then $I : g^\infty = \hat{I} \cap k[x_1, \dots, x_n]$

and if G is a lex GB of $\hat{I}$ for $y > x_1 > \dots > x_n$

$\Rightarrow$ $G \cap k[x_1, \dots, x_n]$ is a basis of $\hat{I} : g^\infty$.

Proof:

(i) $h \in (g) \Rightarrow h = bg$ ← a poly.

Let $f \in \left(\frac{h_1}{g}, \dots, \frac{h_p}{g}\right) \Rightarrow hf = bgf \in (h_1, \dots, h_p)$

$= I \cap (g)$

$\therefore$ $hf \in I \Rightarrow f \in I : g \subseteq I$

Take $f \in I : g \Rightarrow fg \in I$ $\quad fg \in (g)$

$\therefore$ $fg \in I \cap (g) = (h_1, \dots, h_p)$

$\therefore$ $fg = \sum r_i h_i$

but $h_i \in (g) \Rightarrow \frac{h_i}{g}$ is a polynomial

$\therefore$ $f = \sum r_i \left(\frac{h_i}{g}\right) \Rightarrow f \in \left(\frac{h_1}{g}, \dots, \frac{h_p}{g}\right)$

(ii) Homework/midterm

$\blacksquare$

Alg for colon ideal

Input $I=(f_1,\dots,f_r)$, $J=(g_1,\dots,g_s)$

Output $I:J$

DO:

- Compute a basis/GB for $I:g_i$
  - for each of these:
    - compute a GB $(h_1,\dots,h_p)$ of $(f_1,\dots,f_r)\cap(g_i)$ using intersection alg.
    - divide h_j by g_i $\forall j$ (Have basis for $I:g_i$)

Compute basis for $I:J$

For $i=2,\dots,s$ do

$I:(g_1,\dots,g_i)=(I:(g_1,\dots,g_{i-1}))\cap I:g_i$

For saturation

Input $I=(f_1,\dots,f_r)$, $J=(g_1,\dots,g_s)$

Output $I:J^\infty$

- Compute basis for $I:g_i^\infty$ using Thm $\uparrow$ (ii)
- Do intersection loop as above to get $I:(g_1,\dots,g_s)$