Notation : • $I \subseteq k[x_1,\dots,x_n]$ an ideal

• $I_1 = I \cap k[x_2,\dots,x_n]$

$a = (a_2,\dots,a_n) \in V(I_1)$ is a partial solution

Goal : Find $a_1 \in k$ s.t $(a_1, a) \in V(I)$

$f \in k[x_1,\dots,x_n]$ $\qquad N \geq 0$

$f = c_f(x_2,\dots,x_n)x_1^N$ + smaller $\deg_{x_1}$ terms

$\deg_{x_1}(f) = N$

Lemma : Suppose $f = \sum_{j=1}^{t} A_j g_j$ — standard representation w.r.t some G.B. for lex order $x_1 > \dots > x_n$ Then:

(i) $\deg_{x_1}(f) \geq \deg_{x_1}(A_j g_j)$

(ii) $c_f = \sum_{\deg_{x_1}(A_j g_j) = N} c_{A_j} c_{g_j}$ where $N = \deg_{x_1}(f)$.

Thm Let $G = \{g_1,\dots,g_t\}$ be a Gröbner basis of $I \subseteq k[x_1,\dots,x_n]$ w.r.t. lex $x_1 > \dots > x_n$.

Write

$g_j = \underbrace{c_j}_{c_{g_j}}(x_2,\dots,x_n)x_1^{N_j}$ + smaller $\deg_{x_1}$ terms

$c_j \in k[x_2,\dots,x_n]$

Assume $a = (a_2,\dots,a_n) \in V(I_1)$ is a partial sol.

Such that $a \notin V(c_1, \dots, c_t)$. Then

$$(*) \quad \{ f(x_1, a) \mid f \in I \} = (g_0(x_1, a)) \subseteq K[x_1]$$

$g_0 = g \in G$ s.t.

- $\deg_{x_1}(g)$ is the minimum x_1 degree amoung $g \in G$ for which $c_g(a) \neq 0$

Furthmore:

(i) $\deg(g_0(x_1, a)) > 0$

(ii) If $g_0(a_1, a) = 0$ for $a_1 \in K$ then $(a_1, a) \in V(I)$.

Aside) If $I = (f_1, \dots, f_s)$, $V = V(I)$

If $a \in V(I_1)$ is a part. sol. then when we evalate $f_1 = \dots = f_s$ at a this reduces to a single eq, $g_0(x_1, a) = c$ AND we go comes a lex GB of I.

$g_0 = c_0 x_1^{\ell} + \sim$

Proof:

choose an "optimal" $g_0 \in G$. Show $\deg(g_0(x_1, a)) > 0$

Suppose $\deg(g_0(x_1, a)) = 0 \Rightarrow \deg_{x_1}(g_0) = 0$

Since $c_0(a) \neq 0$ (by def) $\Rightarrow g_0 \in I_1$ and $c_0 = g_0$

but since $a \in V(I_1) \Rightarrow g_0(a) = c_0(a) = 0$

this is a contradiction (of $c_0(a) \neq 0$)

$\therefore \deg(g_0(x_1, a)) > 0$

(ii) follow from $(*)$ since if $g_0(a_1, a) = 0$

and $\{ f(x_1, a) \mid f \in I \} = (g_0(x_1, a)) \subseteq k[x_1]$

$\Rightarrow \quad f(a_1, a) = 0 \quad \forall f \in I \quad (a_1, a) \in \mathcal{V}(I).$

Define $\mathrm{Eval}_a : k[x_1, \dots, x_n] \longrightarrow k[x_1]$

$$f(x_1, \dots, x_n) \longmapsto f(x_1, a)$$

This is a ring homomorphism

$\therefore$

- Image of ideal under Eval$_a$ is an ideal
- Image is generated by $g_j(x_1, a)$ $\forall j$

i.e. $\mathrm{Eval}_a(I) = (g_1(x_1, a), \dots, g_t(x_n, a))$

$\therefore$ need to show $g_j(x_1, a) \in (g_0(x_1, a))$ $\forall g_j \in G$.

Two steps:

Step 1: prove $g_j(x_1, a) = 0$ when $g_j \in G$

s.t. $\deg_{x_1}(g_j) < \deg_{x_1}(g_0)$.

Step 2: Prove that $g_j(x_1, a) \in (g_0(x_1, a))$ by induction on $\deg_{x_1}(g_j)$.

Step 1: $d_0 = \deg_{x_1}(g_0)$

- g_0 does not drop x_1 degree under eval. at a since $c_0(a) \neq 0$.
- any $g_j \in G$ with $\deg_{x_1}(g_j) < d_0$ either drops x_1 degree or vanishes under evaluation at a.

Want to obtain a Contradiction

Suppose some $g_j \in G$, $\deg_{x_1}(g_j) < d_0$ with

$g_j(x_1, a) \neq 0 \in k[x_1]$. Among these

g_b = One which minimizes x_1-degree drop when evaluated at a ("bad")

Set $\boxed{\delta = \underbrace{\deg_{x_1}(g_b)}_{= d_b} - \deg(g_b(x_1, a))}$ = Degree drop under eval at a

Any other $g_j \in G$ with $\deg_{x_1}(g_j) < d_0$ either vanishes to $0 \in k[x_1]$ or drops degree by at least δ under eval at a.

Let

$$S = c_0 x_1^{d_0 - d_b} g_b - c_b g_0 \in I$$

(c_0 goes with g_0)

$$= c_0 x_1^{d_0 - d_b}(c_b x_1^{d_b} + \cdots) - c_b \cdot (c_0 x_1^{d_0} + \cdots)$$

$$\therefore \deg_{x_1}(S) < d_0$$

Compute $\deg(S(x_1, a))$ in two ways

Eval at a

$$S(x_1, a) = c_0(a) x_1^{d_0 - d_b} g_b(x_1, a) - \underbrace{c_b(a)}_{=0} g_0(x_1, a)$$

$$= c_0(a) x_1^{d_0 - d_b} g_b(x_1, a)$$

$c_0(a) \neq 0$

$$\therefore \quad \boxed{\begin{aligned} \deg(S(x_1,a)) &= d_0 - d_b + \deg(g_b(x_1,a)) \\ &= d_0 - d_b + d_b - \delta \\ &= d_0 - \delta . \end{aligned}}$$

Now calculate $\deg(S(x_1,a))$ again

use std. rep. in GB $\quad S = \sum_{j=1}^{t} B_j g_j$

$\left(\text{recall} \quad \deg_{x_1}(S) < d_0 \right)$

From Lemma

$$\deg_{x_1}(B_j) + \deg_{x_1}(g_j) = \deg_{x_1}(B_j g_j) \leq \deg_{x_1}(S) < d_0$$

$\uparrow \; B_j \neq 0$

$$\deg_{x_1}(g_j) < d_0$$

$\Rightarrow$ Either $g_j(x_1,a) = 0$ OR $\deg_{x_1} g_j$ drops by at least δ under evaluation

$$\deg(B_j(x_1,a)) + \deg(g_j(x_1,a)) \leq \overbrace{\deg_{x_1}(B_j) + \deg_{x_1}(g_j)}^{\deg < d_0} - \delta$$

$$< d_0 - \delta$$

$\therefore$ Eval at a in $S = \sum B_i g_i$ give

$$\boxed{\begin{aligned}\deg(S(x_1,a)) &\le \max\left(\deg(B_i(x_1,a)) + \deg(g_i(x_1,a))\right)\\ &< d_0 - \delta \quad \text{This is a contradiction}\end{aligned}}$$

$\therefore$ all GB elements with lower x_1 degree than g_0 vanish at a

Step: Do induction on $\deg_{x_1}(g_j)$

to show $g_j(x_1,a) \in (g_0(x_1,a)) \quad \forall g_j \in G$

Base case is when $\deg_{x_1}(g_j) < d_0$

Since by step 1 $\Rightarrow \quad g_j(x_1,a) = 0 \in (g_0(x_1,a))$

Fix $d \ge d_0$ and assume

$g_j(x_1,a) \in (g_0(x_1,a)) \quad \forall g_j$ with $\deg_{x_1}(g_j) < d$

Now take $g_j \in G$ with $\deg_{x_1}(g_j) = d$

Consider

$$S = C_0 g_j - C_j x_1^{d-d_0} g_0 \in I$$

$$= c_0(c_j x_1^d + \cdots) - c_j x_1^{d-d_0}(c_0 x^{d_0} + \cdots)$$

$\Rightarrow \quad \deg_{x_1}(S) < d$

As before take std. rep. of $S = \sum_{l=1}^{t} B_l g_l$

$\Rightarrow \deg_{x_1}(g_l) < d$ when $B_l \neq 0$

By induction $g_l(x_1, a) \in (g_0(x_1, a))$

we have

$$c_0 g_j = c_j x_1^{d-d_0} g_0 + S = c_j x_1^{d-d_0} g_0 + \sum_{l=1}^{t} B_l g_l$$

Evaluate at a

$$c_0(a) g_j(a) = c_j(a) x_1^{d-d_0} \overbrace{g_0(x_1, a)}^{\in (g_0(x_1,a))} + \sum_{l=1}^{t} B_l g_l^{\in (g_0(x_1,a))}$$

Since $c_0(a) \neq 0$

$\therefore \quad g_j(x_1, a) \in (g_0(x_1, a)) \quad \forall j$

Theorem 3 (The Extension Theorem). *Let $I = \langle f_1, \ldots, f_s \rangle \subseteq k[x_1, \ldots, x_n]$ and let I_1 be the first elimination ideal of I. For each $1 \leq i \leq s$, write f_i in the form*

$$f_i = c_i(x_2, \ldots, x_n) x_1^{N_i} + \textit{terms in which } x_1 \textit{ has degree} < N_i,$$

where $N_i \geq 0$ and $c_i \in k[x_2, \ldots, x_n]$ is nonzero. Suppose that we have a partial solution $(a_2, \ldots, a_n) \in \mathbf{V}(I_1)$. If $(a_2, \ldots, a_n) \notin \mathbf{V}(c_1, \ldots, c_s)$ and k is algebraically closed, then there exists $a_1 \in k$ such that $(a_1, \ldots, a_n) \in \mathbf{V}(I)$.

Proof: Let $G = \{g_1, \ldots, g_t\}$ be a lex GB of I $(x_1 > \cdots > x_n)$ set $a = (a_2, \ldots, a_n)$

show $\exists\ g_j \in G$ s.t $c_{g_j}(a) \neq 0$

Since $a \notin V(c_1, \ldots, c_s)$

$\Rightarrow$ $c_i(a) \neq 0$ for some i

Take std. rep. of this $f_i = \sum_{j=1}^{t} A_j g_j$

by Lemma (part 2)

$$c_i = \sum c_{A_j} c_{g_j}$$

$$\deg_{x_1}(A_j g_j) = \deg_{x_1}(f_i)$$

Since $c_i(a) \neq 0$ at least 1 $c_{g_j}(a) \neq 0$