Prop $I \subseteq k[x_1, \dots, x_n] \longleftrightarrow I(V)$ for $V = \{(a_1, \dots, a_n)\} \subseteq k^n$

$I = (x_1 - a_1, \dots, x_n - a_n)$

is a maximal ideal.

Proof: Suppose $J \supsetneq I \Rightarrow \exists\ f \in J$ s.t. $f \notin I$

by div. alg.

$$f = A_1(x_1 - a_1) + \dots + A_n(x_n - a_n) + b$$

$b \in k$

$f \notin I$

$\therefore\ b \neq 0 \qquad f \in J$

$\therefore\ b \in J$

$\Rightarrow 1 \in J \Rightarrow J = k[x_1, \dots, x_n]$

I is maximal $\blacksquare$

$V(x_1 - a_1, \dots, x_n - a_n) = \{(a_1, \dots, a_n)\}$

$\therefore$ Every point in k^n gives a maximal ideal

Converse false over k - not alg. closed

Thm If k alg. closed then every maximal ideal of $k[x_1, \dots, x_n]$ has the form $(x_1 - a_1, \dots, x_n - a_n)$ $a_i \in k$.

Proof: $I \subseteq k[x_1, \dots, x_n]$ maximal. $I \neq k[x_1, \dots, x_n]$

$\therefore\ V(I) \neq \phi$ by weak Nullstellensatz

$\therefore \quad (a_1, \dots, a_n) \in V(I)$ for some $a_i \in k$

$f \in I \Rightarrow f \in I(\{(a_1, \dots, a_n)\})$
$$\| \quad (x_1 - a_1, \dots, x_n - a_n)$$

$\therefore \quad I \subseteq (x_1 - a_1, \dots, x_n - a_n) \subsetneq k[x_1, \dots, x_n]$

$\Rightarrow \quad I = (x_1 - a_1, \dots, x_n - a_n)$ (since maximal) $\blacksquare$

Cor.) k alg. closed. There is a 1-1 correspondence between points of k^n and maximal ideals of $k[x_1, \dots, x_n]$.

Prop) V is irreducible iff for every $W \subsetneq V$ $V \setminus W$ is Zariski dense in V, i.e $\overline{V \setminus W} = V$.

Proof: Take V irr. $W \subsetneq V$ ← a variety

$V = W \cup \overline{V \setminus W}$

V is irr. and $V \neq W$ by def of irriducible

$V = \overline{V \setminus W}$

Now take $V = V_1 \cup V_2$. If $V_1 \subsetneq V$ $\overline{V \setminus V_1} = V$ by asumtion

But $V \setminus V_1 \subseteq V_2 \Rightarrow \overline{V \setminus V_1} \subseteq V_2 \Rightarrow V \subseteq V_2$

$\Rightarrow \quad V = V_2 \quad \therefore \quad V$ is irreducible $\blacksquare$

$S \subseteq V$

Zariski dense if $\bar{S} = V$

$S \subseteq W \subsetneq V$

Decompositions into irr.

Prop 1 (Descending chain con.). DCC

Any decending chain of varieties

$$V_1 \supseteq V_2 \supseteq \cdots \quad \text{in } k^n$$

must stabalize, i.e. $\exists\, N > 0$ s.t. $V_N = V_{N+1} = \cdots$.

Proof) ACC

$$I(V_1) \subseteq I(V_2) \subseteq \cdots$$

Thm) Let $V \subseteq k^n$ be a variety. we can write

$$V = V_1 \cup \cdots \cup V_m$$

where each V_i is an irreducible variety.

Proof:

Assume V can't be written as a finite union of irr ibuble

$\Rightarrow$ V is not irreducible

$\Rightarrow$ $V = V_1 \cup \tilde{V_1}$ $\quad V \neq V_1$ and $V \neq \tilde{V_1}$

$\Rightarrow$ $V_1 = V_2 \cup \tilde{V_2}$ $\quad V_1 \neq V_2, \; V_1 \neq \tilde{V_2}$

V_2 not a finite union of irr...

$\Rightarrow$ we have

$$V \supseteq V_1 \supseteq V_2 \supseteq \cdots\cdots$$

$V \neq V_1 \neq V_2 \neq \cdots\cdots$ This contradicts the D.C.C.

$\blacksquare$

$\therefore$

$$V(xz, yz) = V(x,y) \cup V(z)$$

is this unique

Def) Let $V \subseteq k^n$ a variety. A decomp.

$$V = V_1 \cup \cdots \cup V_m$$

is a <u>minimal decomp.</u> if $V_i \not\subseteq V_j$ for $i \neq j$

Call V_i the <u>irreducible components</u> of V.

Thm) A variety $V \subseteq k^n$ has a minimal decomp.

$$V = V_1 \cup \cdots \cup V_m$$

and this decomp is unique

Proof : we know $V = V_1 \cup \cdots \cup V_{\hat{m}}$ exists for V_i irriducible. If $V_i \subseteq V_j$ $i \neq j$ remove V_i until no such inclusion exists this gives a minimal decomp.

$$V = V_1 \cup \cdots \cup V_m$$

Uniquness :

Suppose $V = \hat{V}_1 \cup \cdots \cup \hat{V}_\ell$ is also a minimal decomp

$$V_i = V_i \cap V = V_i \cap (\hat{V}_1 \cup \cdots \cup \hat{V}_\ell)$$

$$= (V_i \cap \hat{V}_1) \cup \cdots \cup (V_i \cap \hat{V}_\ell)$$

V_i is irreducible $\therefore$ $V_i = V_i \cap \hat{V}_j$ for some j

$$V_i \subseteq \hat{V}_j$$

Same procedure gives $\hat{V}_j \subseteq V_k$

$\therefore$ $V_i \subseteq \hat{V}_j \subseteq V_k$ $\therefore$ $V_i = \hat{V}_j = V_k$ by minimality.