Ex) $I = (x^2+y-1,\ xy-2y^2+2y) \subseteq k[x,y]$

lex GB gives $\langle LT(I)\rangle = (x^2, xy, y^3)$

A vector space basis for $k[x,y]/I$

is $\{1, x, y, y^2\}$

$$\overset{\in k[x,y]/I}{f \cdot g} = (3y^2+x)\cdot(x-y) = \underset{\uparrow \text{ Not st. rep}}{3xy^2+x^2-3y^2-xy}$$

$$\overline{f\cdot g}^G = -\frac{11}{4}y^2 - \frac{5}{4}y + 1$$

Proposition 5. *Let I be an ideal in $k[x_1,\ldots,x_n]$ and let G be a Gröbner basis of I with respect to any monomial order. For each $[f] \in k[x_1,\ldots,x_n]/I$, we get the standard representative $\overline{f} = \overline{f}^G$ in $S = \mathrm{Span}(x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I)\rangle)$. Then:*

(i) *$[f]+[g]$ is represented by $\overline{f}+\overline{g}$.*

(ii) *$[f]\cdot[g]$ is represented by $\overline{\overline{f}\cdot\overline{g}} \in S$.*

$k[V]$

$I = I(V)$

↑ Do your ops. in $k[V]$ and reduce w.r.t a GB of I.

Thm) (Finiteness Thm) Fix a monomial ordering $I \subseteq k[x_1, \dots, x_n]$ an ideal.

(i) For each i $1 \le i \le n$ $\exists\, m_i \ge 0$ s.t $x_i^{m_i} \in (LT(I))$

(ii) a a G B for I

this is same

$$x^{m_i} = LM(g) \text{ for some } g \in G$$

(iii) $\{ x^\alpha \mid x^\alpha \notin (LT(I)) \}$ is finite

(iv) $\dim_k \left(k[x_1, \dots, x_n]/I \right) < \infty$

(v) If k alg closed $V(I) \subseteq k^n$ is a finite set.

(i) – (iv) are equivalent $\forall k$. (i) throug (v) are equivalent for k. alg closed

Proof:

(iii) $\Leftrightarrow$ (iv) From Prop.

(i) $\Leftrightarrow$ (ii) ch.2.

(i) $\Leftrightarrow$ (iii)

Show (i) $\Rightarrow$ (iii)

If $x_i^{m_i} \in (LT(I))$ for each i

then the $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for which some $\alpha_i \ge m_i$ are all in $(LT(I))$

$\therefore$ The monomials in complement of $(LT(I))$
must have $0 \le d_i \le m_i - 1 \quad \forall i$

$\therefore$ number of monomials in complement
is at $m_1 \cdots m_n$

(iii) $\Rightarrow$ (i) Suppose complement has $N < \infty$ monomials
$\Rightarrow$ for each i at least 1 of $1, x_i, \ldots, x_i^N$
is in $(LT(I))$

(iv) $\Rightarrow$ (v)

$$\dim_K (K[x_1, \ldots, x_n] / I) < \infty$$

Show $\exists$ finitely many distinct ith
coords of points of V.

Fix an arbitrary i.
Finite dim.
Consider $[x_i^j] \in K[x_1, \ldots, x_n]/I \qquad j = 0, 1, \ldots$
family of coord powers.

$\therefore [x_i^j]$ must be lin dependent

$\exists c_j$ (not all zero) and m s.t

$$[0] = \sum_{j=0}^{m} c_j [x_i^j] = \left[\sum c_j x_i^j \right]$$

$\therefore \quad \sum c_j x_i^j \in I$

$$\Rightarrow \left(\sum c_j^{(1)} x_1^j, \ldots, \sum C_j^{(n)} x_n^j \right) \subseteq I$$

a 1-var poly can have only a finite number of solutions

$$\Rightarrow V(I) \subseteq V\left(\sum c_j^{(1)} x_1^j, \ldots, \sum C^{(n)}_j x_n^j \right)$$

$\therefore \; V(I)$ is finite.

Assume k is alg. closed show $(v) \Rightarrow (i)$

$V(I)$ is finite $\Rightarrow$ If $V = \emptyset$ then $1 \in I$ by weak Nullstelentatz $\therefore \; x_i^0 \in (LT(I)) \; \forall i$

If V is non-empty, fix i, let $a_1, \ldots, a_m \in k$ be the distinct i^{th} of points in V

Consider $f(x_i) = \prod_{j=1}^{m} (x_i - a_j)$

$$\Rightarrow f \in I(V)$$

By Nullstel. $\Rightarrow f^N \in I$

$$\Rightarrow LM(f^N) = x_i^{mN} \in (LT(I)) \quad \blacksquare$$

Prop Let $I \subseteq k[x_1,\dots,x_n]$ be an ideal s.t. $\dim_k(k[x_1,\dots,x_n]/I) < \infty$ (and $\Leftrightarrow$ $x_i^{m_i} \in (LT(I))$ for some m_i $\forall i$)

(i) # points in $V \leq \dim_k(k[x_1,\dots,x_n]/I)$

(ii) # points in $V \leq m_1 \cdots m_n$

(iii) If I radical, k alg. closed

$$\dim_k(k[x_1,\dots,x_n]/I) = \#\text{ points in } V.$$

Proof (sketch)

- Show given distinct points $p_1,\dots,p_m \in k^n$ $\exists$ $f_1,\dots,f_m$ s.t. $f_i(p_i) = 1$ and $f_i(p_j) = 0$ $\forall i \neq j$

V is finite, $V = \{p_1,\dots,p_m\}$ distinct

prove that $[f_1],\dots,[f_m] \in k[x_1,\dots,x_n]/I$ are lin. independent

[Suppose $a_i \in k$ $\sum a_i[f_i] = [0]$

$\Rightarrow g = \sum a_i f_i \in I$

$\therefore 0 = g(p_i) = a_i$]