Prop Let $V \subseteq k^n$ be an aff. variety. The following are equivalent:

(i) V is irreducible

(ii) $I(V)$ is a prime ideal

know from ch 4

(iii) $k[V]$ is an integral domain

Proof: (iii) $\Rightarrow$ (i) Suppose $k[V]$ is an integral domain

$$k[V] = \mathrm{Hom}(V, k)$$

Suppose V is reducible

$$V = V_1 \cup V_2$$

proper non-empty subvar.

$\exists$ f_1 s.t. $f_1(p) = 0$ $\forall p \in V_1$ and $f_1(q) \neq 0$ for some $q \in V_2$

and f_2 s.t. $f_2(q) = 0$ $\forall q \in V_2$, $f_2(p) \neq 0$ for some $p \in V_1$

$\therefore$ $f_1 \neq 0$, $f_2 \neq 0$ in $k[V]$

$$f_1 \cdot f_2 = 0 \text{ in } k[V]$$

$\Rightarrow$ $k[V]$ is not an integral domain

this is a contradiction

$\therefore$ $k[V]$ int. dom. $\Rightarrow$ V is irreducible

(i) $\Rightarrow$ (iii) If $k[V]$ were not an int. d

$\exists$ poly f, g s.t. $f(p) \neq 0$, $g(q) \neq 0$ for some $p, q \in V$

but $(f \cdot g)(p) = 0 \quad \forall p \in V$

Can u write

$$V = (V \cap V(f)) \cup (V \cap V(g))$$

$\uparrow$ if $f(p) \neq 0 \Rightarrow g(p) = 0$

since $f(p) g(p) = 0 \;\; \forall p \in V$

$\therefore$ we get all points in V

but $V \cap V(f) \subsetneq V$ since $f(p) \neq 0$ for some $p \in V$

$\Rightarrow V$ is reducible.

$\therefore$ int domain $\Rightarrow$ irreducible $\blacksquare$

Prop Distinct poly functions $\phi : V \to k$ are in 1-1 corespondence with equivilence classes in $k[x_1, \dots, x_n]/I(V)$

Proof sketch! If $\phi : p \to f(p)$

adding any poly in $g \in I(V)$ won't change ϕ

$\phi : p \mapsto (f+g)(p)$ by previous prop.

$\Updownarrow$

$f \simeq f+g \mod I$

$\blacksquare$

Cor (of Hilbert basis thm) Every ideal in $k[x_1, \dots, x_n]/I$ is finitely generated.

Thm $k[V] \cong k[x_1, \dots, x_n]/I(V)$

proof $\Phi : k[x_1, \dots, x_n]/I(V) \to k[V]$

$[f] = \phi$

where $\phi : V \to k$

$p \mapsto f(p)$

Check that this is a hom.

$\Phi([f+g]) = \phi$

where $\phi : p \mapsto (f+g)(p)$

$\phi : p \mapsto f(p) + g(p)$

$\Phi([f]) + \Phi([g])$

$k[V] \cong k[x_1, \ldots, x_n]/I(V)$ is called the coordinate ring of a variety

$[x_i] \in k[V]$ is called a coordinate function since, for $p \in V$, $p \mapsto p_i$. The coordinate functions generate $k[V]$ since any poly map is a lin combo of powers of $[x_i]$

Note that if $f = 0$ in $k[x_1, \ldots, x_n]/I(V)$

$\Rightarrow f \in I(V) = f(p) = 0$

Proposition 1. *Fix a monomial ordering on $k[x_1, \ldots, x_n]$ and let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. As in Chapter 2, §5, $\langle \mathrm{LT}(I) \rangle$ will denote the ideal generated by the leading terms of elements of I.*

$f = r \bmod I \quad r = \sum c_\alpha x^\alpha$

(i) *Every $f \in k[x_1, \ldots, x_n]$ is congruent modulo I to a unique polynomial r which is a k-linear combination of the monomials in the complement of $\langle \mathrm{LT}(I) \rangle$.*

$x^\alpha \notin \langle \mathrm{LT}(I) \rangle$

(ii) *The elements of $\{x^\alpha \mid x^\alpha \notin \langle \mathrm{LT}(I) \rangle\}$ are "linearly independent modulo I," i.e., if we have*

$$\sum_\alpha c_\alpha x^\alpha \equiv 0 \bmod I,$$

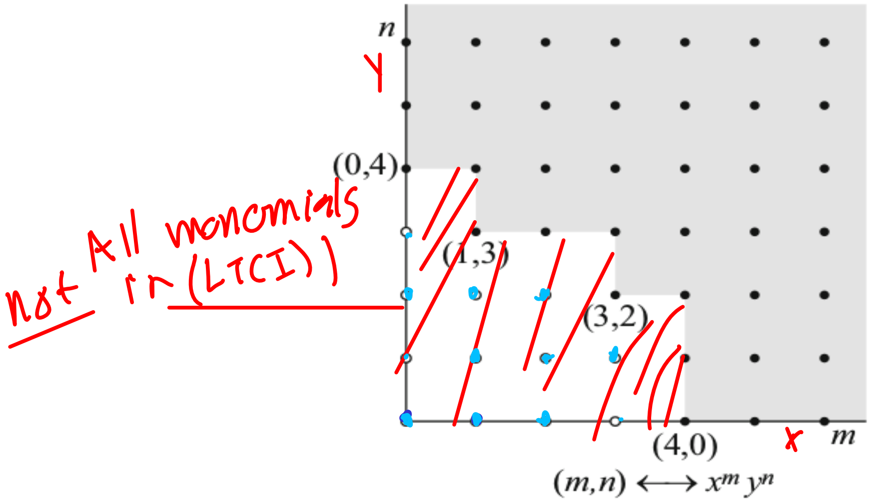
$c_\alpha \in k$

where the x^α are all in the complement of $\langle \mathrm{LT}(I) \rangle$, then $c_\alpha = 0$ for all α.

[Ex] $I = (xy^3 - x^2, x^3y^2 - y)$ in Grlex

$G = \{x^3y^2 - y, x^4 - y^2, xy^3 - x^2, y^4 - xy\}$

$(LT(I)) = (x^3y^2, x^4, xy^3, y^4)$



For any $g \in k[x,y]$ $\bar{g}^G$ is a k-linear combo of

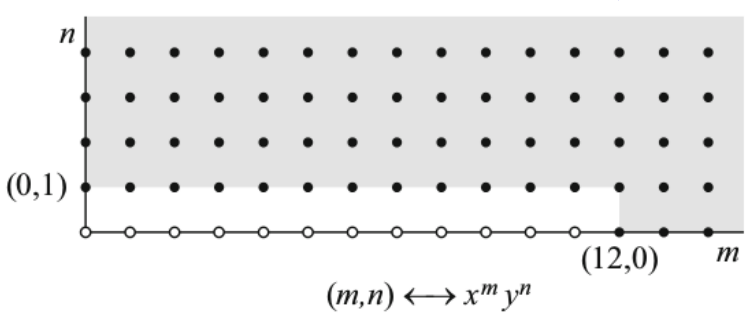
$1, x, x^2, x^3, y, y^2, y^3, xy, x^2y, x^3y,$
y^2x, y^2x^2

$\Big|$
12

reverse lex
i.e. $y > x$

$G = \{ y - x^7, x^{12} - x^2 \}$

$(LT(I)) = (y, x^{12})$



$\therefore$ every $\bar{f}^G \in \text{span}(1, x, \dots, x^{11})$

Prop) Let I be an ideal. Then

$$k[x_1,\dots,x_n]/I \cong \mathrm{Span}_k\left(x^\alpha \mid x^\alpha \notin (LT(I))\right)$$

$\uparrow$ as a k-vector space

Proof) By Prop. $\Phi : k[x_1,\dots,x_n]/I \to S$

$[f] \mapsto \overline{f}^G$

is 1-1 and onto. need to show its a hom.

use $\overline{f}^G$ as a standard rep for $[f]$

recall $\overline{f+g}^G = \overline{f}^G + \overline{g}^G$

if $\overline{f}^G = \sum c_\alpha x^\alpha$, $\overline{g}^G = \sum d_\alpha x^\alpha$

$x^\alpha \notin (LT(I))$

$$\Rightarrow \overline{f+g}^G = \overline{f}^G + \overline{g}^G = \sum (c_\alpha + d_\alpha) x^\alpha$$

$$\therefore \quad \Phi([f+g]) = \Phi([f]) + \Phi([g])$$

Scalar mult follows since $\overline{c\cdot f}^G = c\cdot \overline{f}^G$ ∎