Thm Every ideal $I \subseteq k[x_1, \dots, x_n]$ can be written as a finite intersection of primary ideals

Proof: call an ideal I irreducible if $I = I_1 \cap I_2 \Rightarrow$ either $I = I_1$ or $I = I_2$. Every ideal is the intersection of finitely many irreducibles

[Since: we could construct non-stabilizing accending chain $I \subseteq I_1 \subseteq I_2 \subseteq \cdots$
$I = I_1 \neq I_2 \subseteq \cdots$
otherwise]

by ACC.

Now show an irreducible ideal is primary

Suppose I irr., $fg \in I$, $f \notin I$

$$I : g^\infty = I : g^N \quad \text{for some } N \text{ (large)}$$

$$(I + (g^N)) \cap (I + (f)) = I$$

(check: idea is for this N, $fg^N \in I$)

Since I is irreducible $\Rightarrow$ $I = I + (g^N)$

or $I = I + (f)$
Not this since $f \notin I$

$\Rightarrow$ $I = I + (g^N)$ $\Rightarrow$ $g^N \in I$ $\Rightarrow$ I is primary

∎

Def A <u>primary decomposition</u> of an Ideal I is

$$I = \bigcap_{i=1}^{r} Q_i$$

$\uparrow$ primary ideals

It is minimal (or irredundant) if $\sqrt{Q_i}$ are distinct and $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$

Lemma I, J $\swarrow$ in $k[x_1, \dots, x_n]$ primary, $\sqrt{I} = \sqrt{J} \Rightarrow I \cap J$ is primary

Thm (Lasker-Noether decomp. thm)

Every ideal $I \in k[x_1, \dots, x_n]$ has a minimal primary decomposition

Proof: By Thm above $\exists\, Q_i$ s.t. $I = \bigcap_{i=1}^{r} Q_i$

Suppose $\sqrt{Q_i} = \sqrt{Q_j}$ for some $i \neq j$

$\Rightarrow Q = Q_i \cap Q_j$ is primary $\therefore$ replace Q_i, Q_j by Q

Suppose $Q_i \supseteq \bigcap_{i \neq j} Q_j \Rightarrow$ throw away Q_i

this gives a minimal decomp. $\blacksquare$

Lemma) I primary $\sqrt{I} = P$, $f \in k[x_1, \dots, x_n]$

then

- $f \in I \Rightarrow I : f = (1) = k[x_1, \dots, x_n]$
- $f \notin I \Rightarrow I : f$ is P-primary ideal
- $f \notin P \Rightarrow I : f = I$

Thm) Let $I = \bigcap_{i=1}^{r} Q_i$ be a minimal primary decomp. and $P_i = \sqrt{Q_i}$. Then P_i are precisely the proper prime ideals in

$$\{ \sqrt{I:f} \mid f \in k[x_1, \dots, x_n] \}$$

Closure Thm

Thm (Closure Thm part 2) k alg. closed $V = V(I) \subseteq k^n$. There exists an affine variety $W \subsetneq V(I)$ s.t

$$V(I_\ell) \setminus W \subseteq \pi_\ell(V) \text{ and } \overline{V(I_\ell) \setminus W} = V(I_\ell)$$

Notation: $k[x_1, \dots, x_\ell, y_{\ell+1}, \dots, y_n] = k[x, y]$

Fix an Elimination order i.e. $x^\alpha > x^\beta \Rightarrow x^\alpha > x^\beta y^\gamma$ $\forall \gamma$

i.e. Lex with $x_1 > \dots x_\ell > y_{\ell+1} > \dots > y_n$

$$A \setminus B = A \setminus (A \cap B)$$

Theorem 2. *Fix a field k. Let $I \subseteq k[\mathbf{x}, \mathbf{y}]$ be an ideal and let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for I with respect to a monomial order as above. For $1 \le i \le t$ with $g_i \notin k[\mathbf{y}]$, write g_i in the form*

$$(1) \qquad g_i = c_i(\mathbf{y})\mathbf{x}^{\alpha_i} + \text{terms} < \mathbf{x}^{\alpha_i}.$$

Finally, assume that $\mathbf{b} = (a_{l+1}, \ldots, a_n) \in \mathbf{V}(I_l) \subseteq k^{n-l}$ is a partial solution such that $c_i(\mathbf{b}) \neq 0$ for all $g_i \notin k[\mathbf{y}]$. Then:

(i) *The set*

$$\overline{G} = \{g_i(\mathbf{x}, \mathbf{b}) \mid g_i \notin k[\mathbf{y}]\} \subseteq k[\mathbf{x}]$$

is a Gröbner basis of the ideal $\{f(\mathbf{x}, \mathbf{b}) \mid f \in I\}$.

(ii) *If k is algebraically closed, then there exists $\mathbf{a} = (a_1, \ldots, a_l) \in k^l$ such that $(\mathbf{a}, \mathbf{b}) \in V = \mathbf{V}(I)$.*

Proof: (i) Skip

(ii) Note that $\overline{g_i}$ is non-constant $\forall i$ (since $g_i \notin k[y]$)

$\Rightarrow 1 \notin \overline{I}$ ← I evaluated at b $\quad \therefore \overline{I} \subsetneq k[x]$ ← proper

$\therefore$ by Nullstellensatz $V(\overline{I})$ is non-empty

$\therefore \exists\, a \in k^l$

s.t. $a \in V(\overline{I}) = V(\overline{G})$

$\Rightarrow \overline{g_i}(a) = 0 \quad \forall \overline{g_i} \in \overline{G}$

$\therefore g_i(a, b) = 0 \quad \forall i$ since if $g_i \in G \setminus k[y]$ and b is partial sol then $g_i(a,b) = 0$

if $g_i \in k[y] \Rightarrow g_i(b) = g_i(a,b) = 0$

$\therefore (a, b) \in V = V(I)$ ∎

Cor: $V(I_\ell) \setminus V\left(\prod_{g_i \in G \setminus k[y]} c_i(y)\right) \subseteq \pi_\ell(V)$

$A \setminus B = A \setminus (A \cap B)$

This means that

$$W = V(I_\ell) \cap V\left(\prod_{g_i \in G \setminus k[y]} c_i(y)\right) \subseteq V(I_\ell)$$

is s.t.

$$V(I_\ell) \setminus W \subseteq \pi_\ell(V)$$

V Zariski dense in W if $\overline{V} = W$

$\therefore$ If W is Zariski dense in $V(I_\ell)$ we have the closure theorem (pt. II)

W

Proposition 4. *Assume that k is algebraically closed and the Gröbner basis G is reduced. If $\mathbf{V}(I_l) \setminus \mathbf{V}\left(\prod_{g_i \in G \setminus k[\mathbf{y}]} c_i\right)$ is not Zariski dense in $\mathbf{V}(I_l)$, then there is some $g_i \in G \setminus k[\mathbf{y}]$ whose c_i has the following two properties:*

(i) $V = \mathbf{V}(I + \langle c_i \rangle) \cup \mathbf{V}(I : c_i^\infty)$.

(ii) $I \subsetneq I + \langle c_i \rangle$ and $I \subsetneq I : c_i^\infty$.

This shows that if $V(I_\ell) \setminus V\left(\prod_{g_i \in G \setminus k[y]} c_i\right)$ fails to be dense in $V(I_\ell)$ $\Rightarrow$ V can be decomposed into two peices

Proposition 5. *Let k be algebraically closed. Suppose that a variety $V = \mathbf{V}(I)$ can be written $V = \mathbf{V}(I^{(1)}) \cup \mathbf{V}(I^{(2)})$ and that we have varieties*

index not power

$$W_1 \subseteq \mathbf{V}(I_l^{(1)}) \quad \textit{and} \quad W_2 \subseteq \mathbf{V}(I_l^{(2)})$$

such that $\overline{\mathbf{V}(I_l^{(i)}) \setminus W_i} = \mathbf{V}(I_l^{(i)})$ and $\mathbf{V}(I_l^{(i)}) \setminus W_i \subseteq \pi_l(\mathbf{V}(I^{(i)}))$ for $i = 1, 2$. Then $W = W_1 \cup W_2$ is a variety contained in V that satisfies

$$\overline{\mathbf{V}(I_l) \setminus W} = \mathbf{V}(I_l) \quad \textit{and} \quad \mathbf{V}(I_l) \setminus W \subseteq \pi_l(V).$$

Proposition 6 (Maximum Principle for Ideals). *Given a nonempty collection of ideals $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ in a polynomial ring $k[x_1,\ldots,x_n]$, there exists $\alpha_0 \in \mathcal{A}$ such that for all $\beta \in \mathcal{A}$, we have*

$$I_{\alpha_0} \subseteq I_\beta \Longrightarrow I_{\alpha_0} = I_\beta.$$

In other words, I_{α_0} is maximal with respect to inclusion among the I_α for $\alpha \in \mathcal{A}$.

The "Biggest" element of the collection

Now w Prove closure Thm. part II

Proof: work by contradiction

Suppose $\exists$ ideal $I \subseteq k[x_1,\ldots,x_n]$ where closure thm. fails i.e. there is no affine variety $W \subsetneq V(I)$ s.t

$$\underbrace{V(I_\ell)\setminus W \subseteq \pi_\ell(V(I))}_{\text{Always works}} \text{ and } \underbrace{\overline{V(I_\ell)\setminus W} = V(I_\ell)}_{\text{This must fail}}$$

Among the collection of bad ideals (where thm. fails) $\exists$ a biggest one (by maximum principle)

i.e. $\exists$ I s.t theorem fails for I but holds for every larger J, i.e. $\forall J$ s.t. $I \subsetneq J$ closure thm. holds for J

Fix this largest I By Cor.

$$V(I_\ell)\setminus V\Big(\prod_{g_i\in G\setminus k[y]} c_i(y)\Big) \subseteq \pi_\ell(V(I)) \quad \overset{=V}{}$$

Since closure Thm fails for I

$\Rightarrow$

$$\overline{V(I_\ell) \setminus V\Big(\prod_{g_i \in a[k[y]]} c_i(y)\Big)} \subsetneq V(I_\ell)$$

$\uparrow$ not Zariski dense

$\Rightarrow \exists\, i$ s.t.s $\quad I \subsetneq I^{(1)} = I + (c_i(y))$

$I \subsetneq I^{(2)} = I : c_i^\infty$

and $\quad V(I) = V(I^{(1)}) \cup V(I^{(2)})$

By our choice of I (from max. principle)

$I^{(1)}$ and $I^{(2)}$ must satisfy the closure theorem

$\therefore \exists\, W_i \subseteq V(I_\ell^{(i)})$ $\quad i = 1, 2$ s.t. this is W_i for closure thm

$\therefore$ by Prop. $W = W_1 \cup W_2 \subseteq V(I)$

and is the W we want for I

This is a contradiction of our choice of I

$\therefore$ All ideals satisfy closure thm. $\blacksquare$